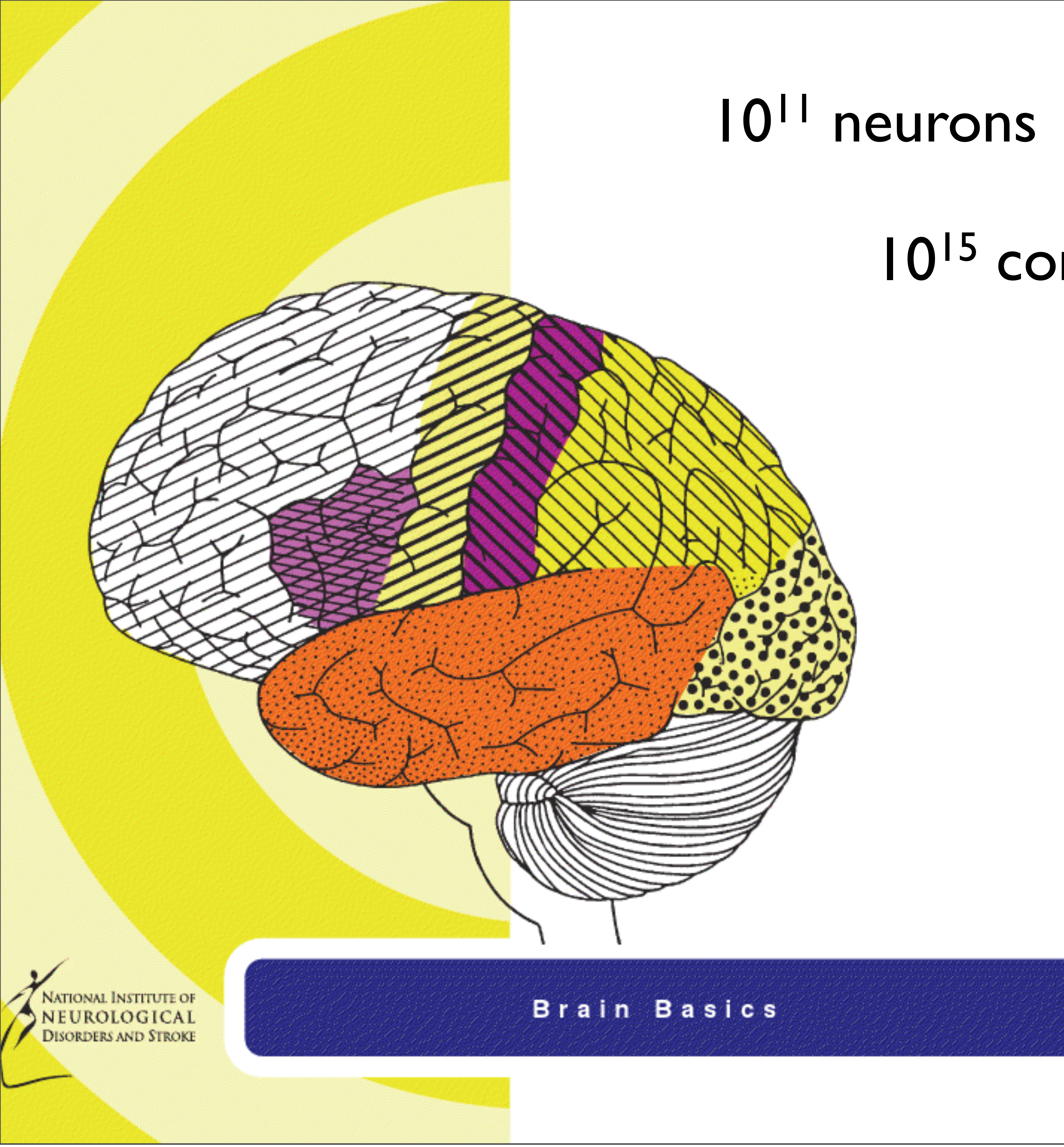
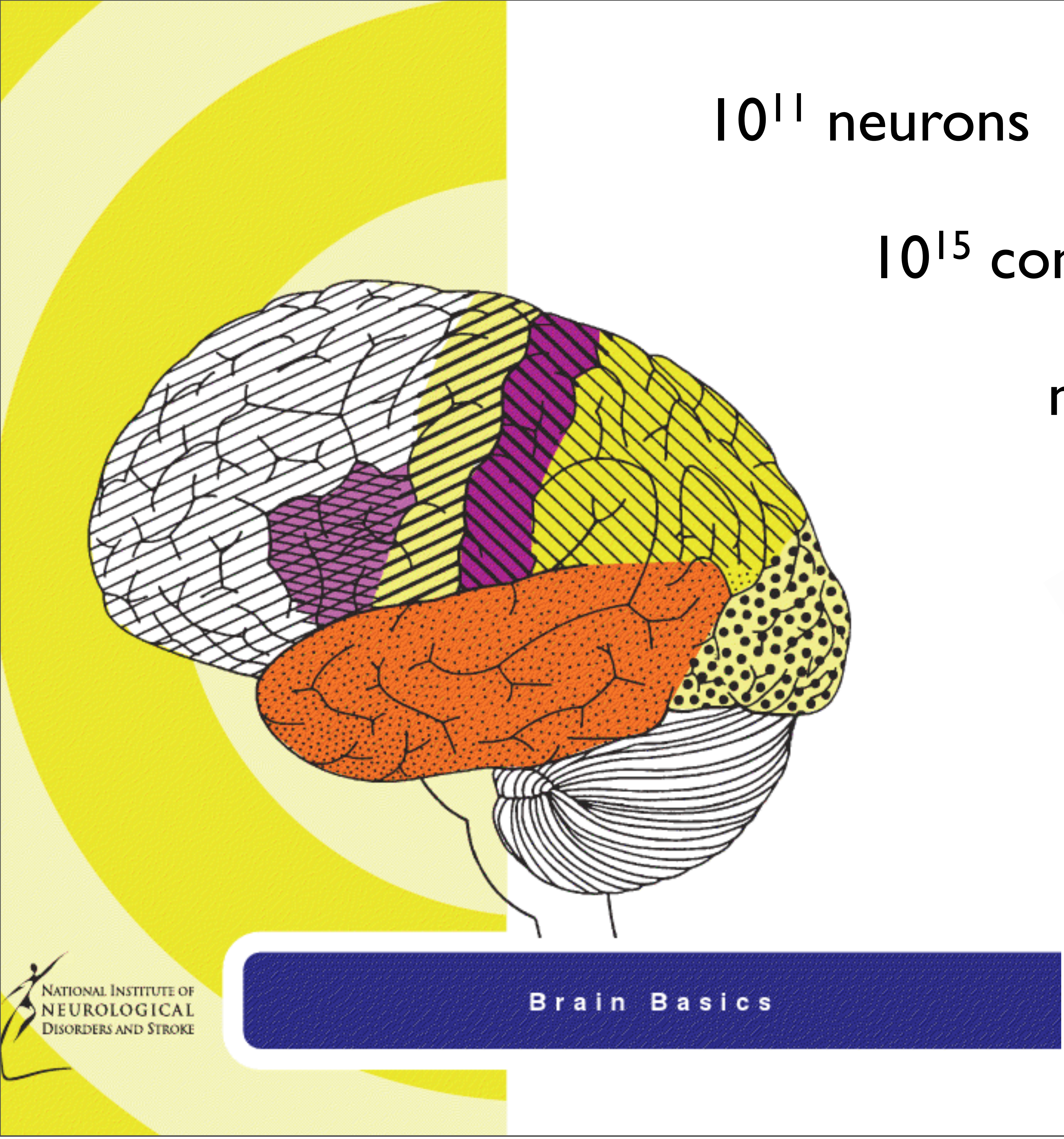
Beyond mean field theory for neural networks

Michael Buice, Carson Chow

10^{11} neurons

10^{15} connections

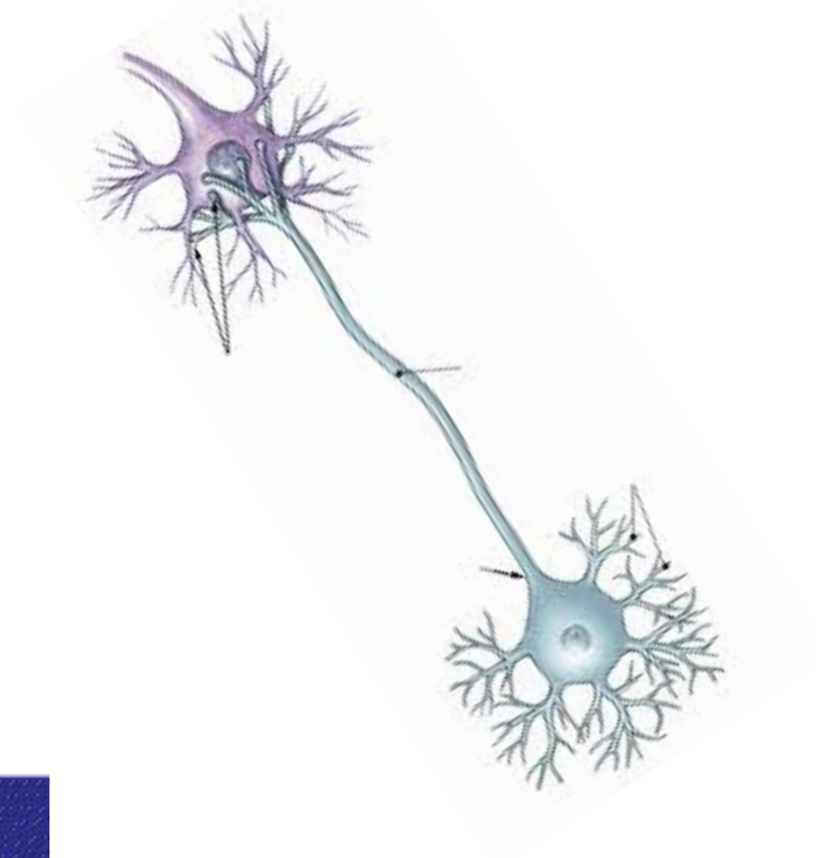




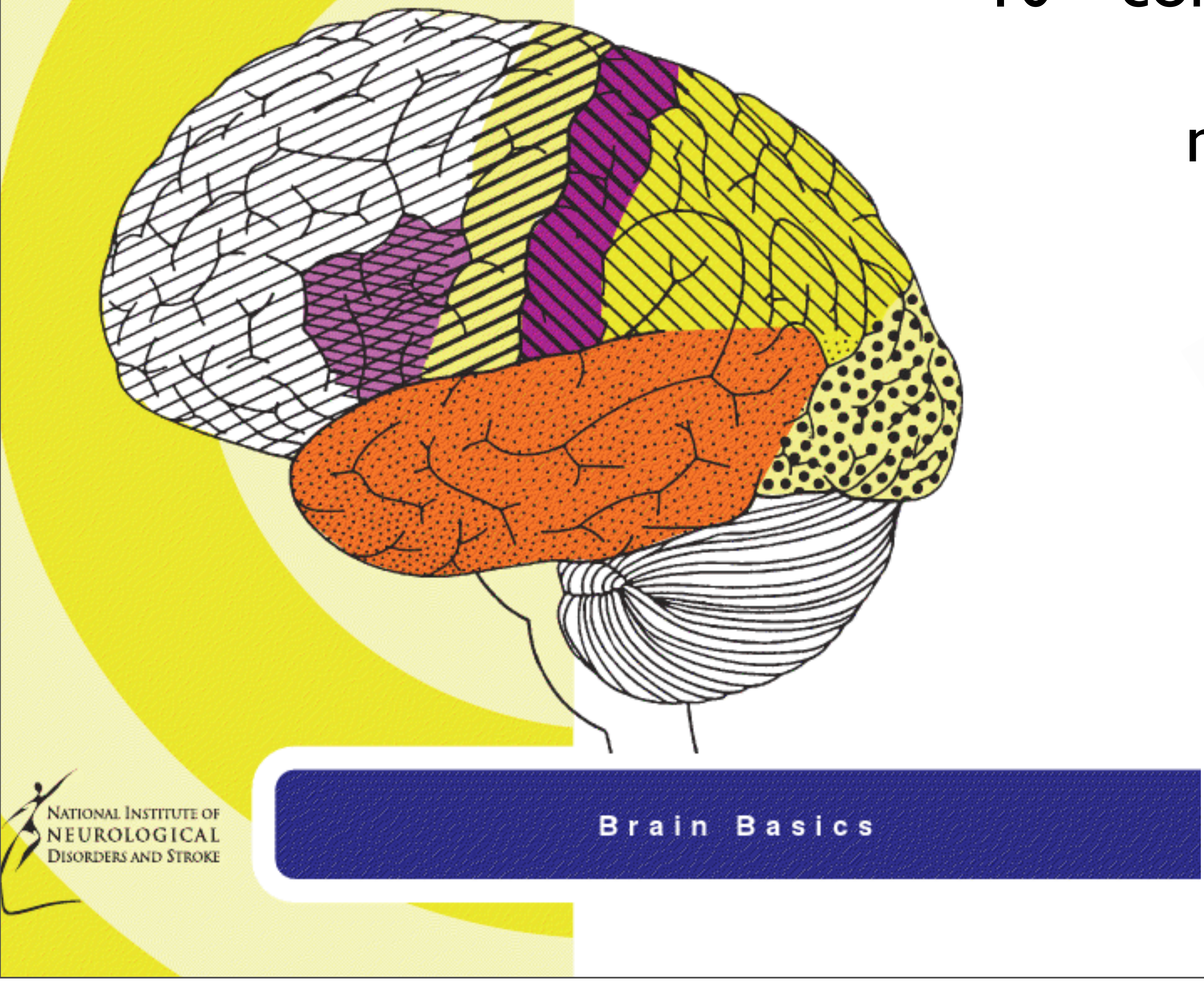
10^{11} neurons

10^{15} connections

neurons are complex



Need to pick a battleground



10^{11} neurons

10^{15} connections

neurons are complex

How do you get complex behavior (e.g. thinking) from the collective action of simple elements (e.g. point neurons)?

Function vs Mechanism

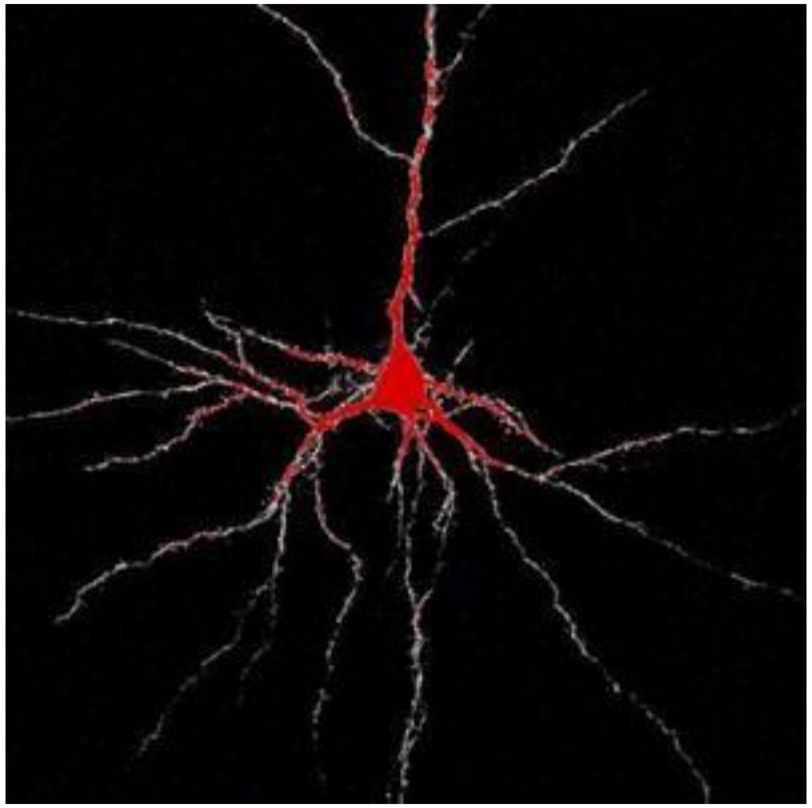
How does the brain do X?

e.g. learning, memory, classification

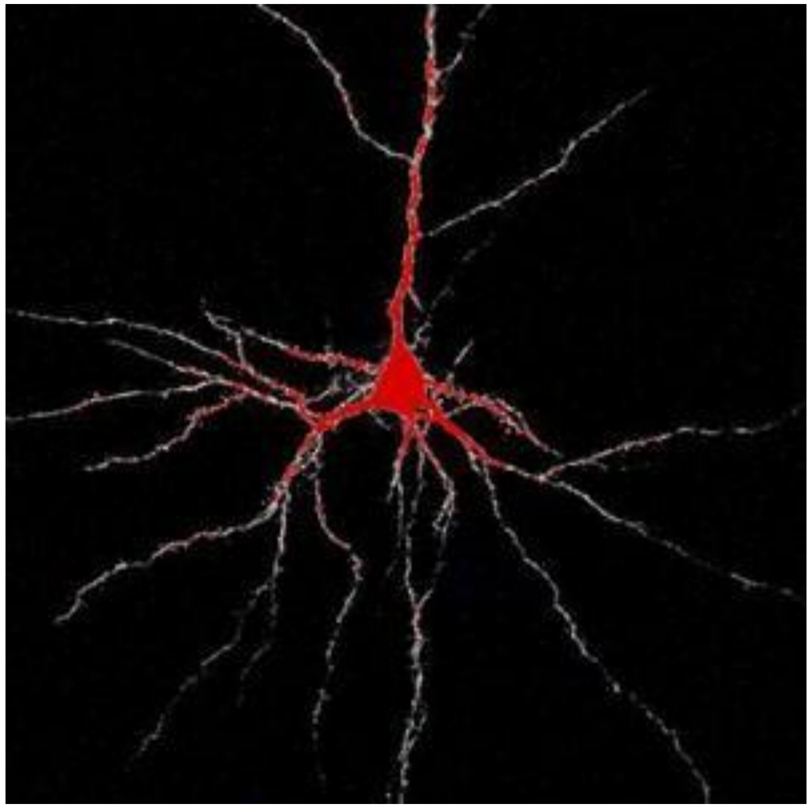
How is X generated in the brain?

e.g. oscillations, synchrony, persistent activity

Neuron



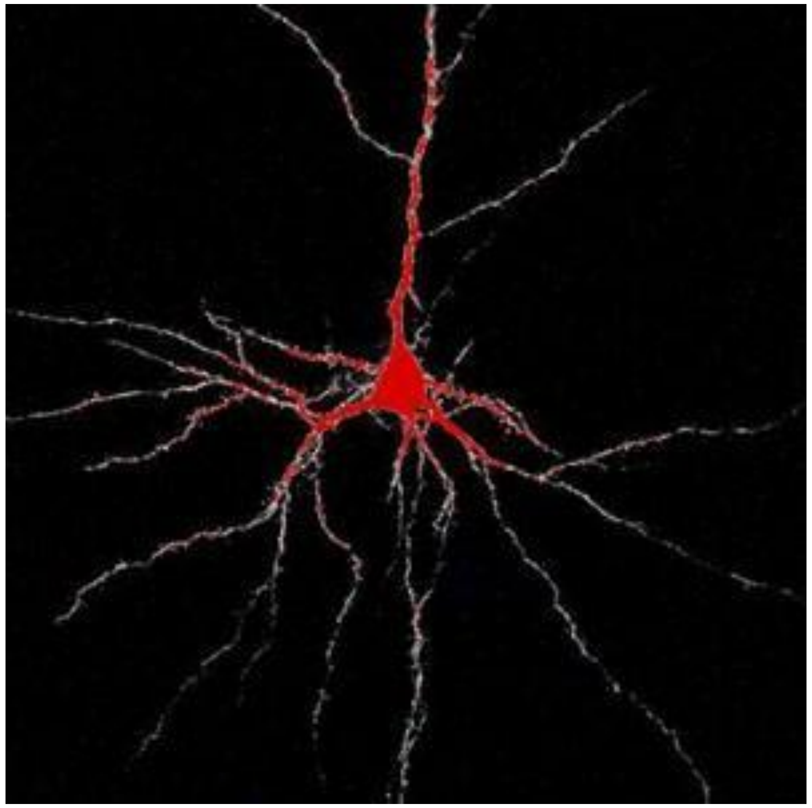
Neuron



$$C\frac{dV}{dt} = -\sum_{r=1}^{n} g_r(x_r)(V - v_r)$$

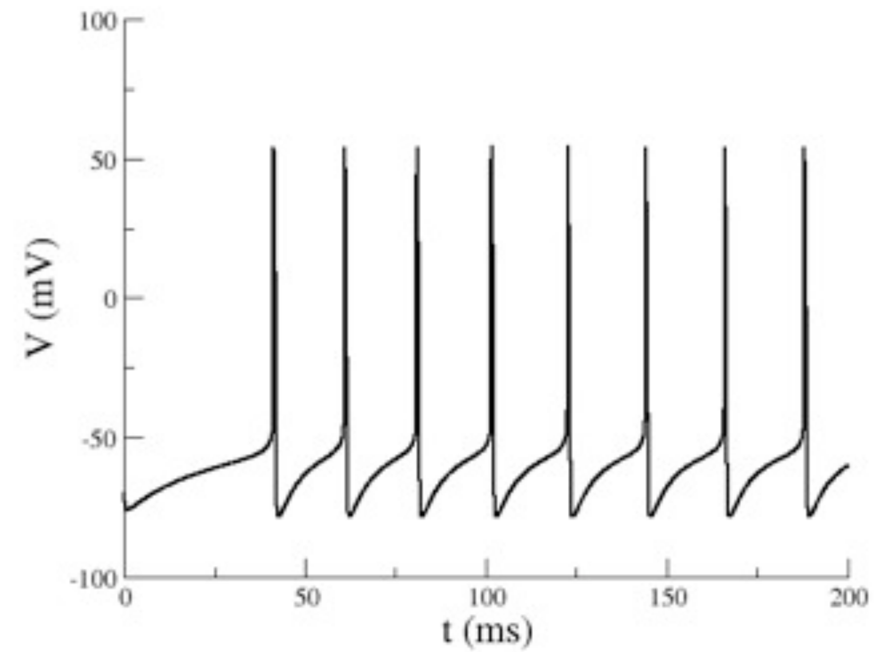
$$\tau_r \frac{dx_r}{dt} = f(x, V) - x_r$$

Neuron

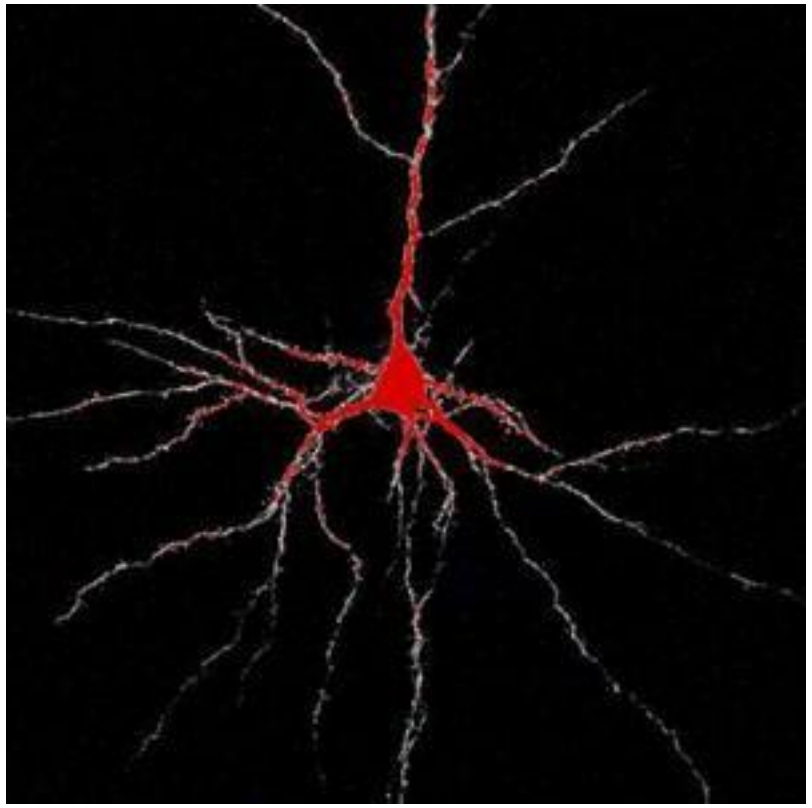


$$C\frac{dV}{dt} = -\sum_{r=1}^{n} g_r(x_r)(V - v_r)$$

$$\tau_r \frac{dx_r}{dt} = f(x, V) - x_r$$

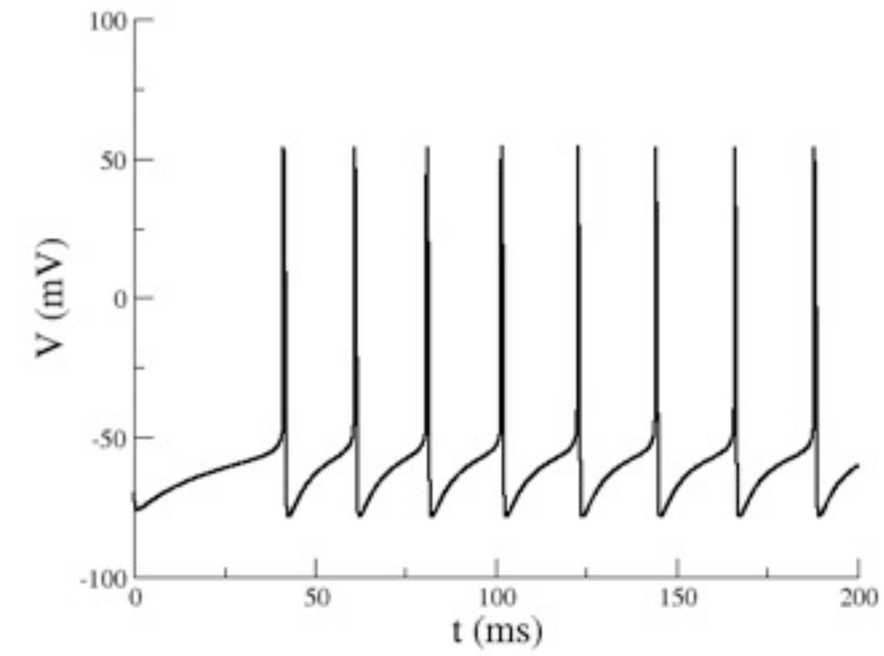


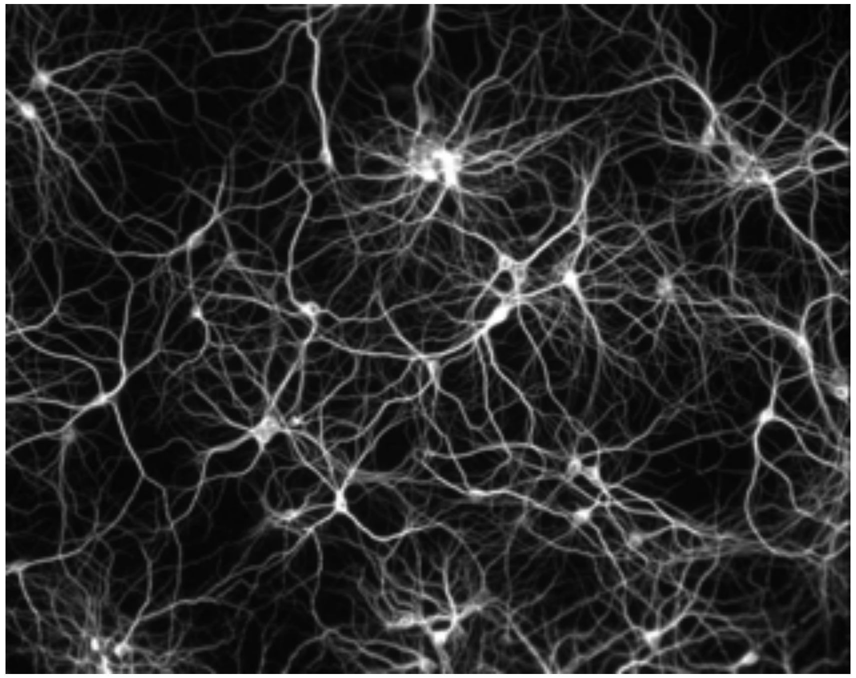
Network



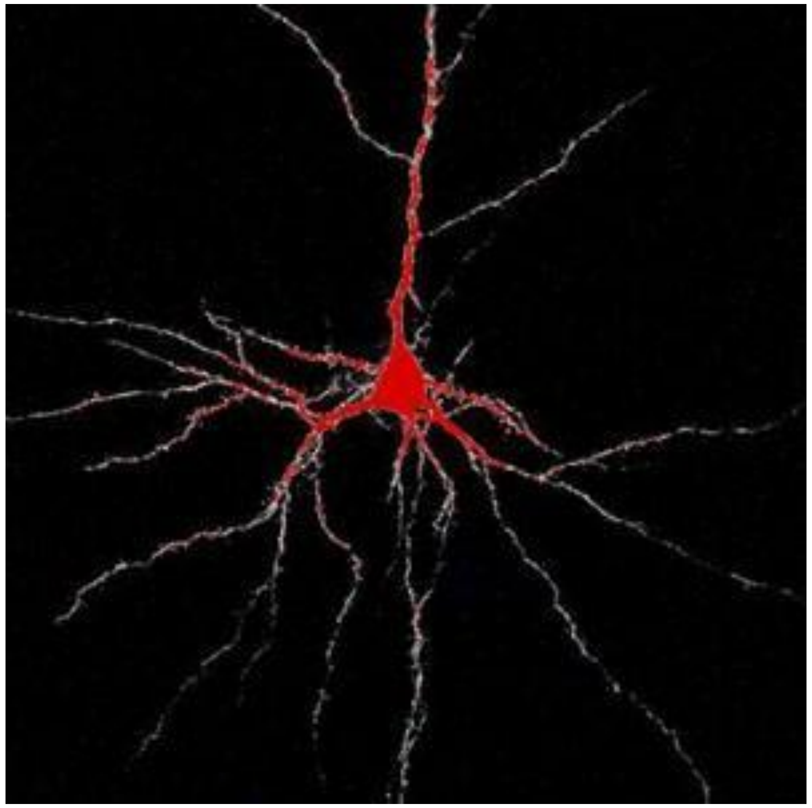
$$C\frac{dV}{dt} = -\sum_{r=1}^{n} g_r(x_r)(V - v_r)$$

$$\tau_r \frac{dx_r}{dt} = f(x, V) - x_r$$



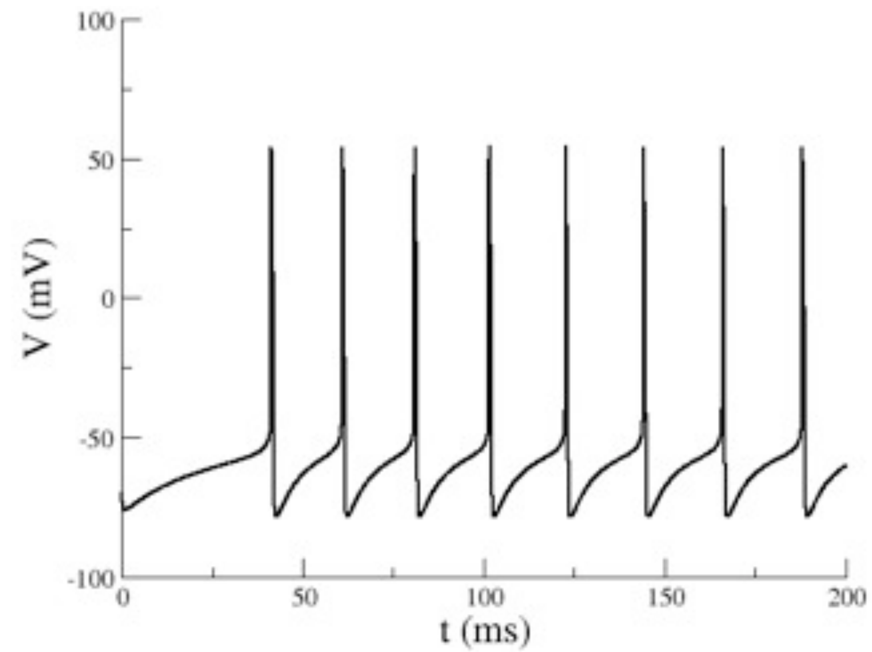


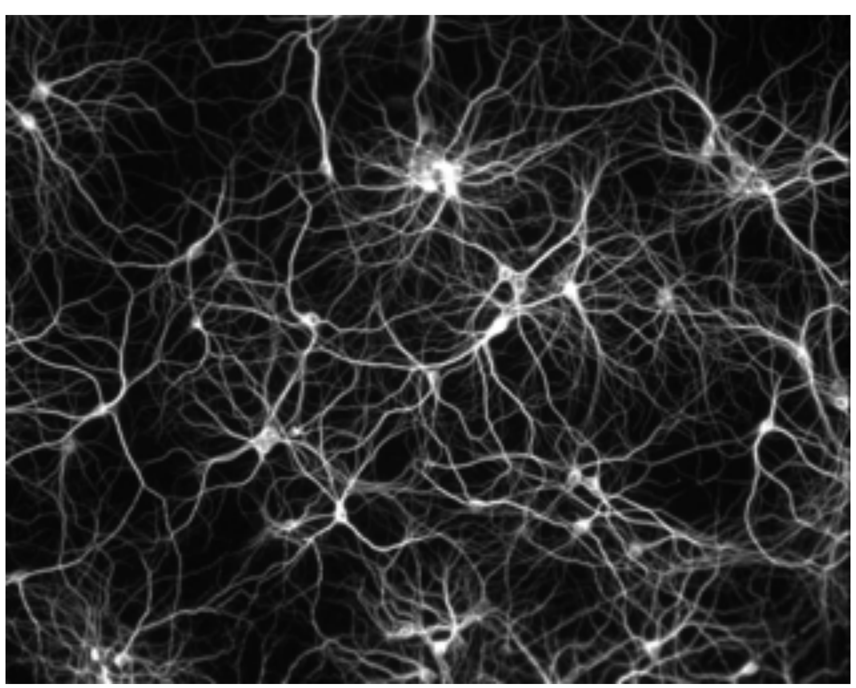
Network



$$C\frac{dV}{dt} = -\sum_{r=1}^{n} g_r(x_r)(V - v_r)$$

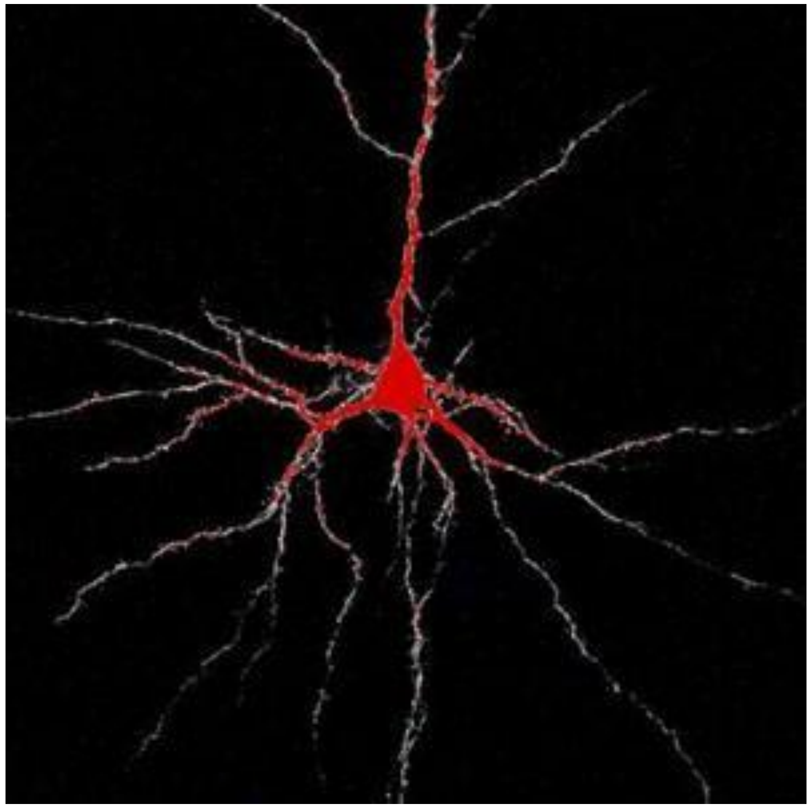
$$\tau_r \frac{dx_r}{dt} = f(x, V) - x_r$$





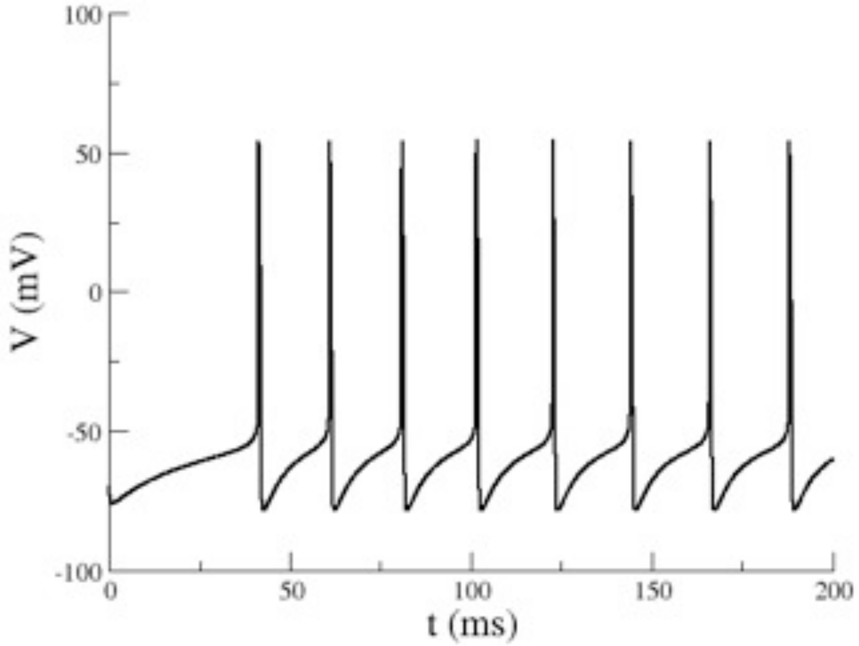
$$C\frac{dV_i}{dt} = -\sum_{r=1}^{n} g_r(x_i^r)(V_i - v_r) + \sum_{j=i}^{N} g_{ij} s_j(t)$$

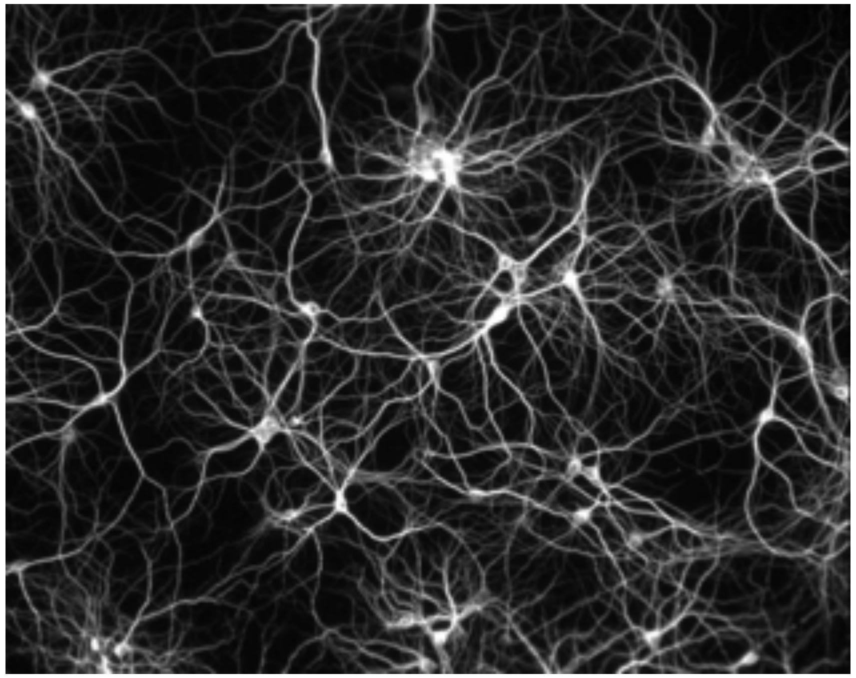
Network



$$C\frac{dV}{dt} = -\sum_{r=1}^{n} g_r(x_r)(V - v_r)$$

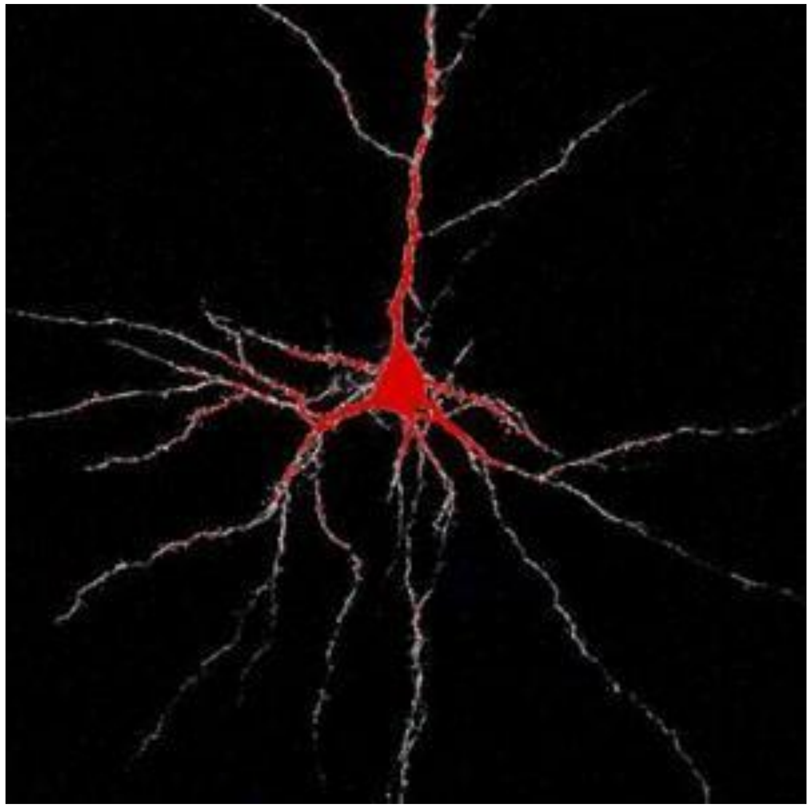
$$\tau_r \frac{dx_r}{dt} = f(x, V) - x_r$$





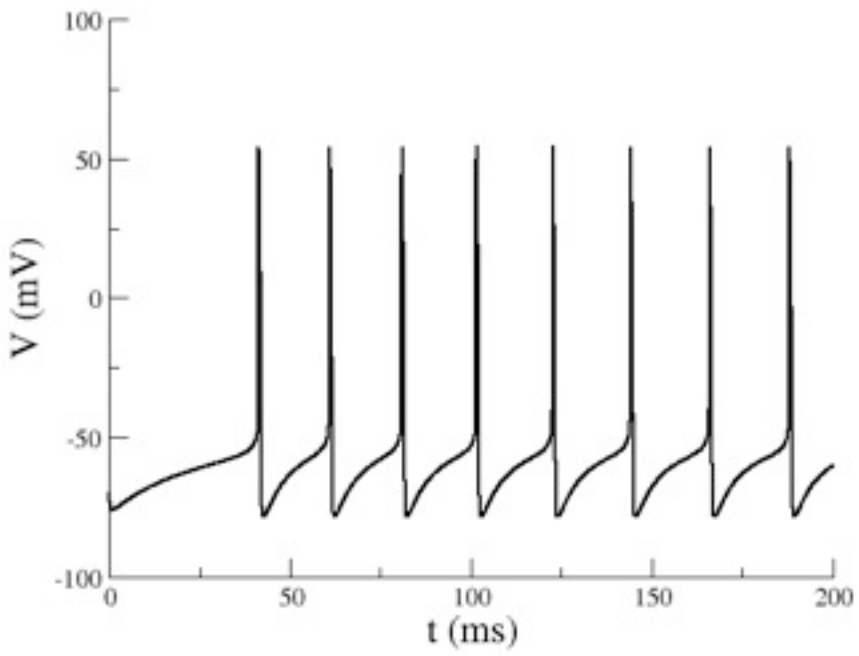
$$C\frac{dV_i}{dt} = -\sum_{r=1}^{n} g_r(x_i^r)(V_i - v_r) + \sum_{j=i}^{N} g_{ij} s_j(t)$$

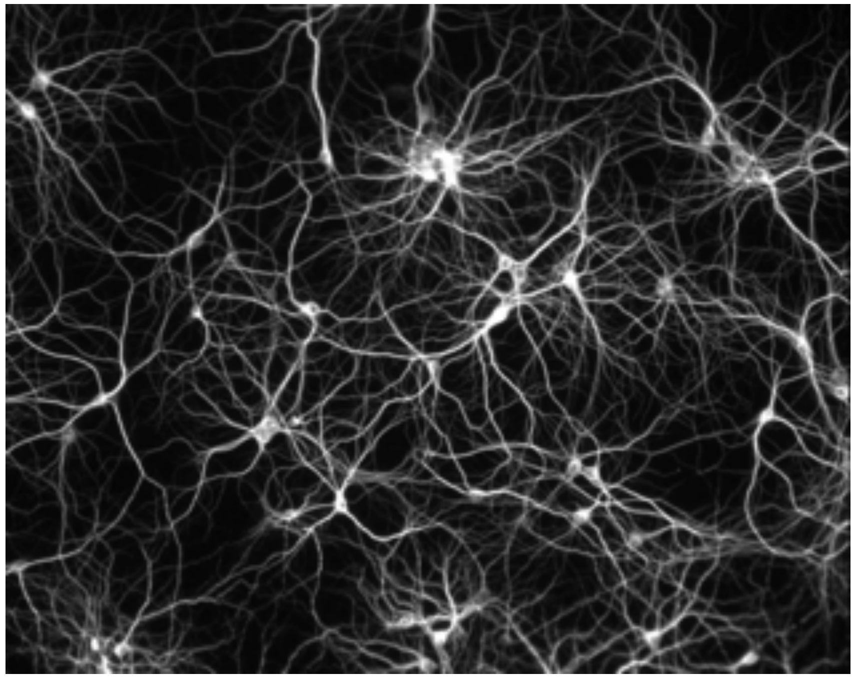
Network



$$C\frac{dV}{dt} = -\sum_{r=1}^{n} g_r(x_r)(V - v_r)$$

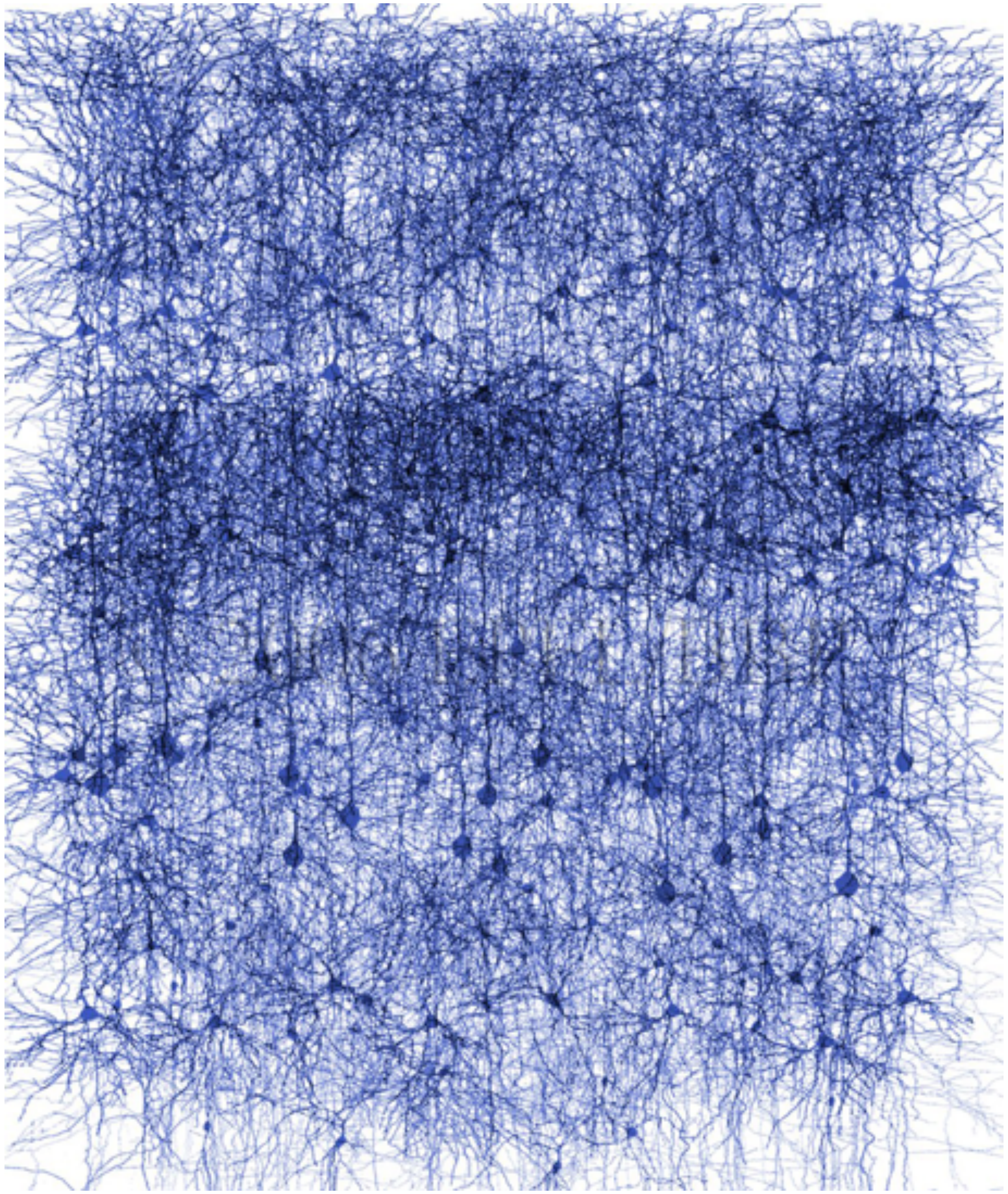
$$\tau_r \frac{dx_r}{dt} = f(x, V) - x_r$$

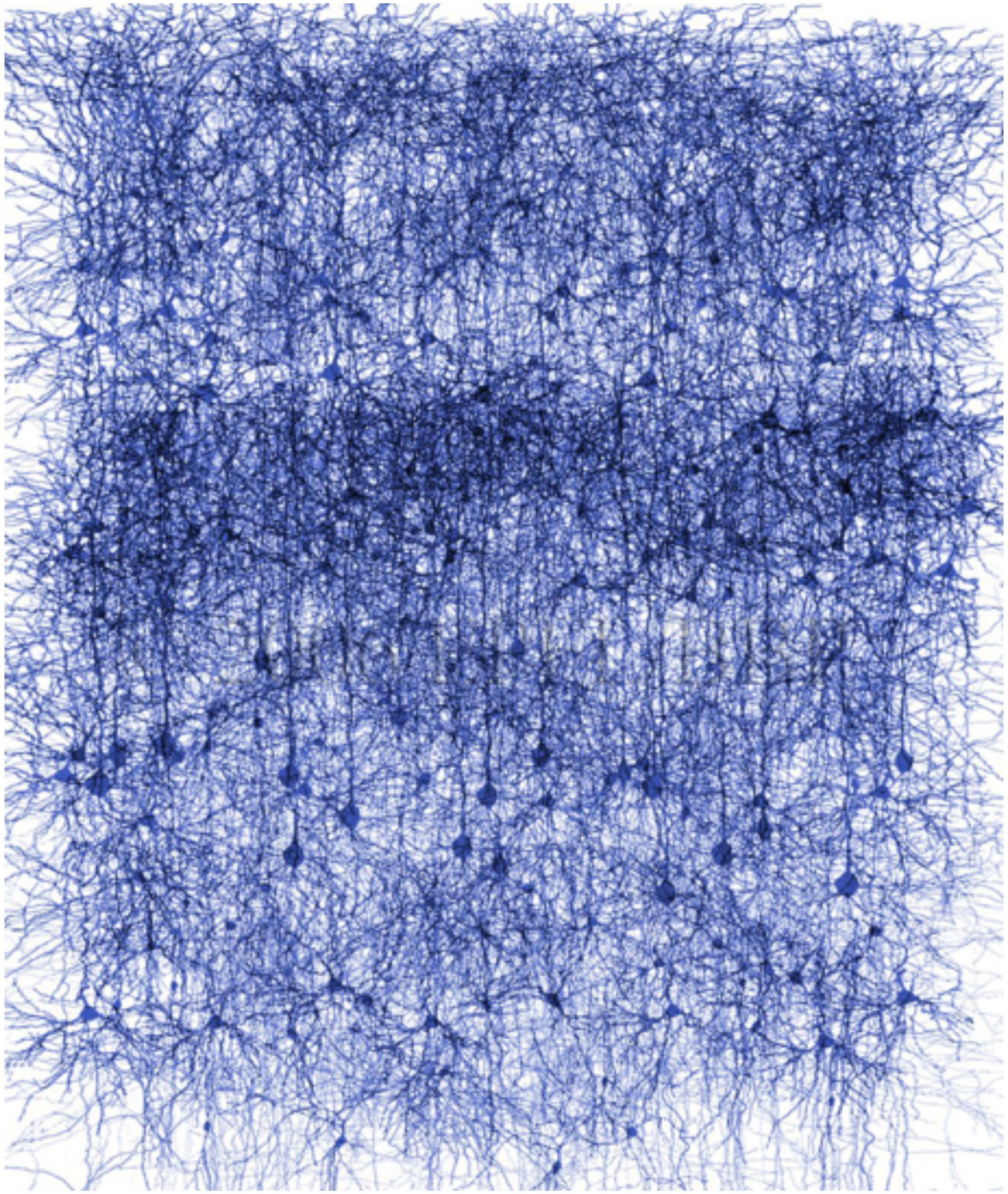




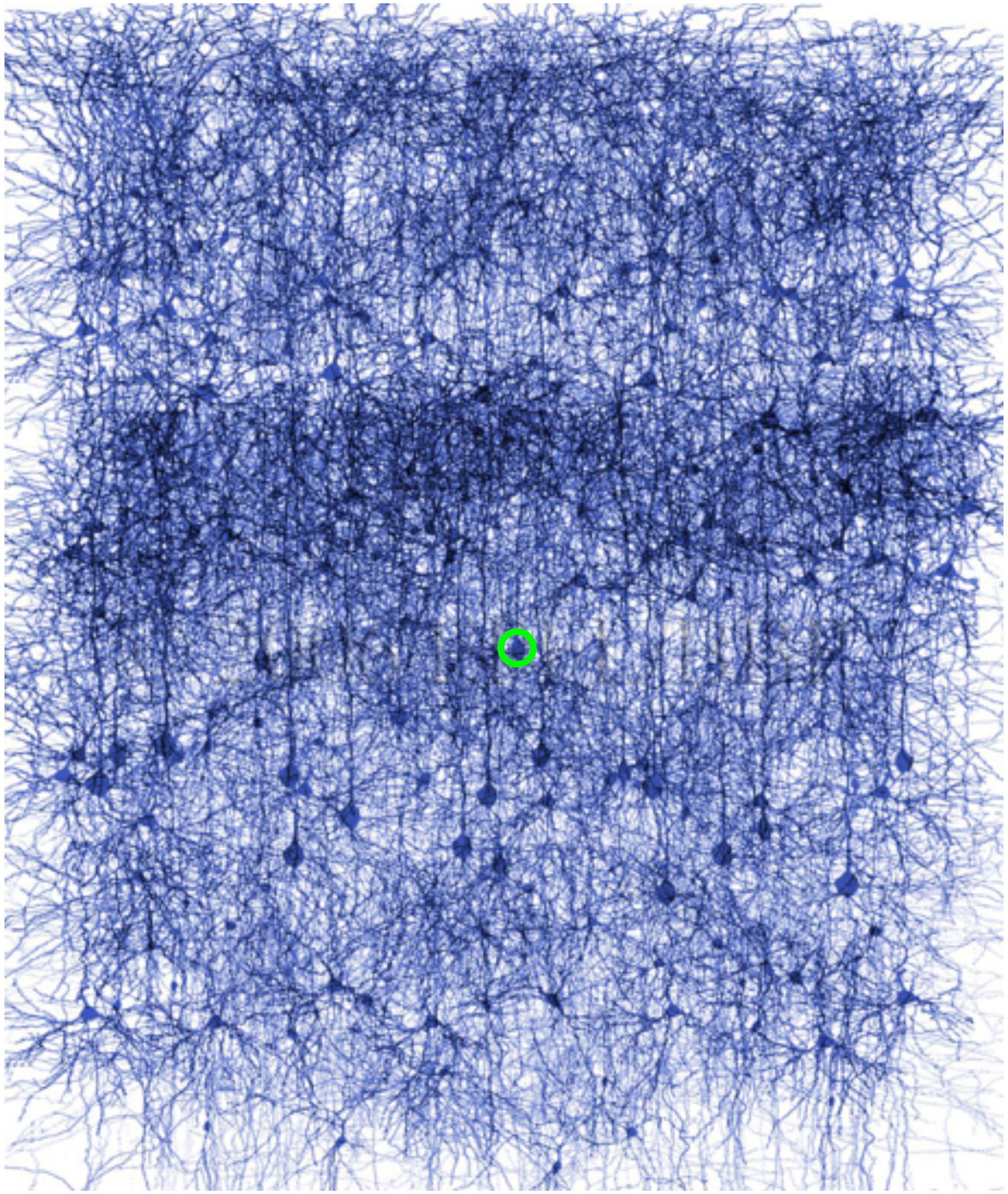
$$C\frac{dV_i}{dt} = -\sum_{r=1}^{n} g_r(x_i^r)(V_i - v_r) + \sum_{j=i}^{N} g_{ij} s_j(t)$$

Really hard

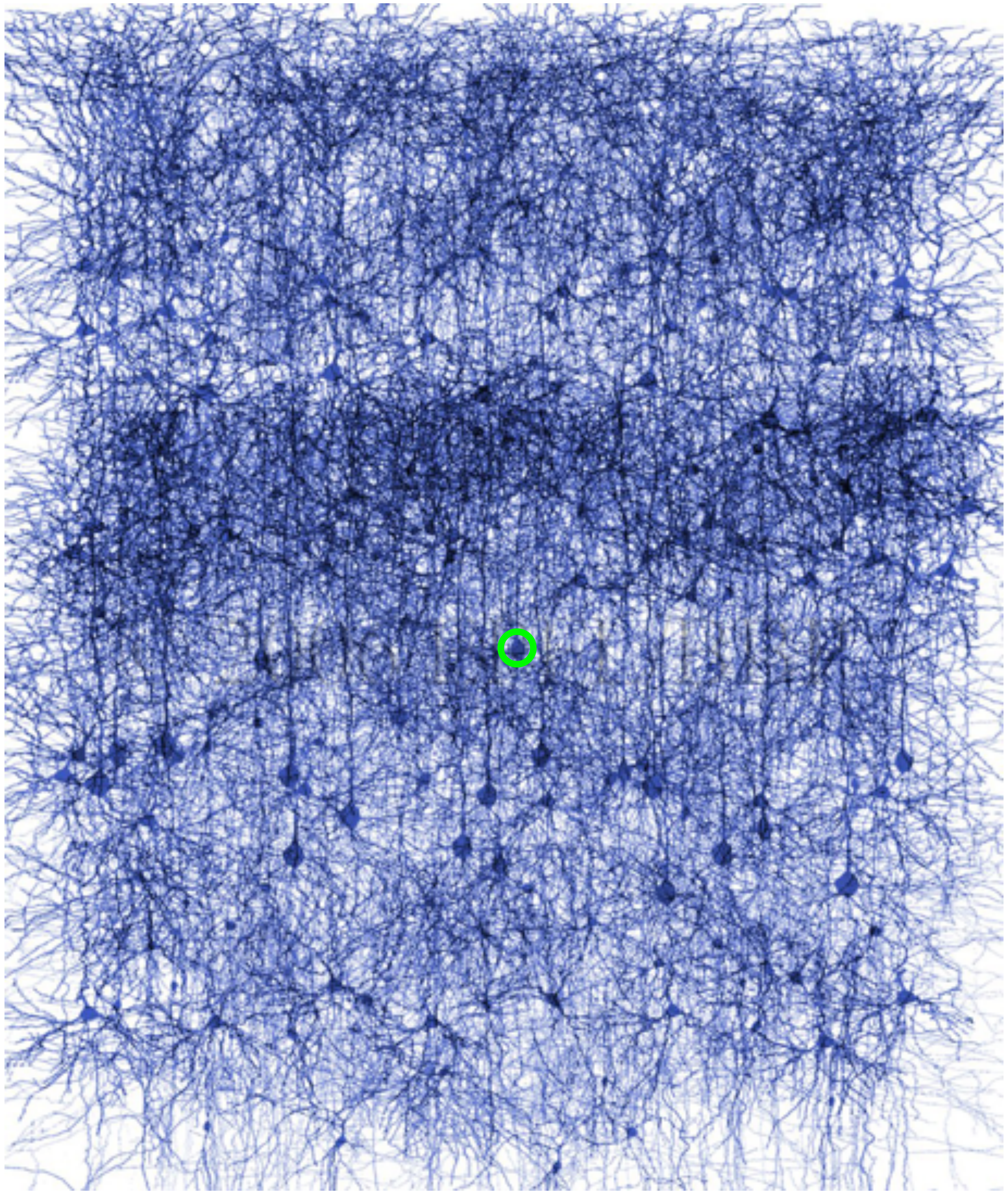


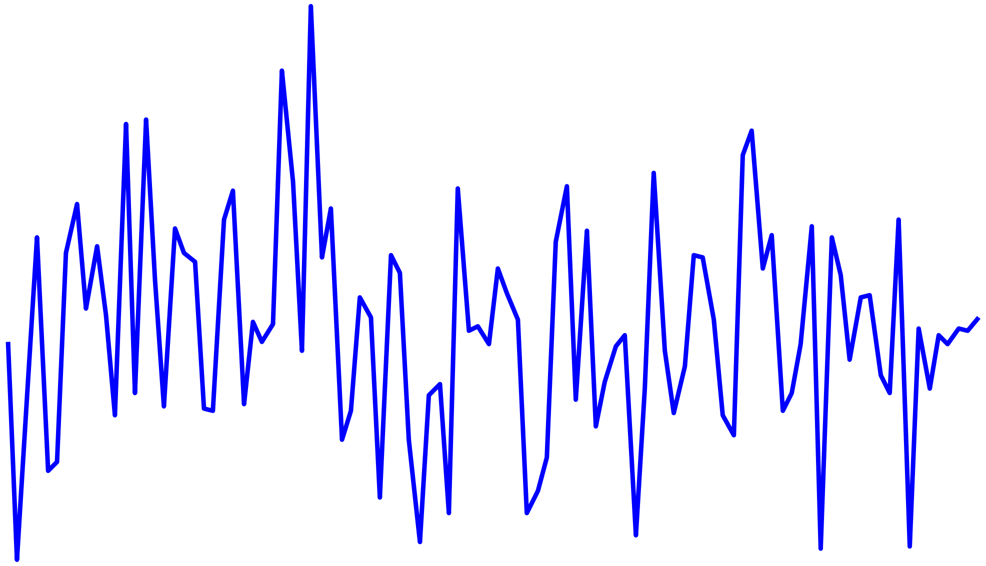


Microscopic → Macroscopic

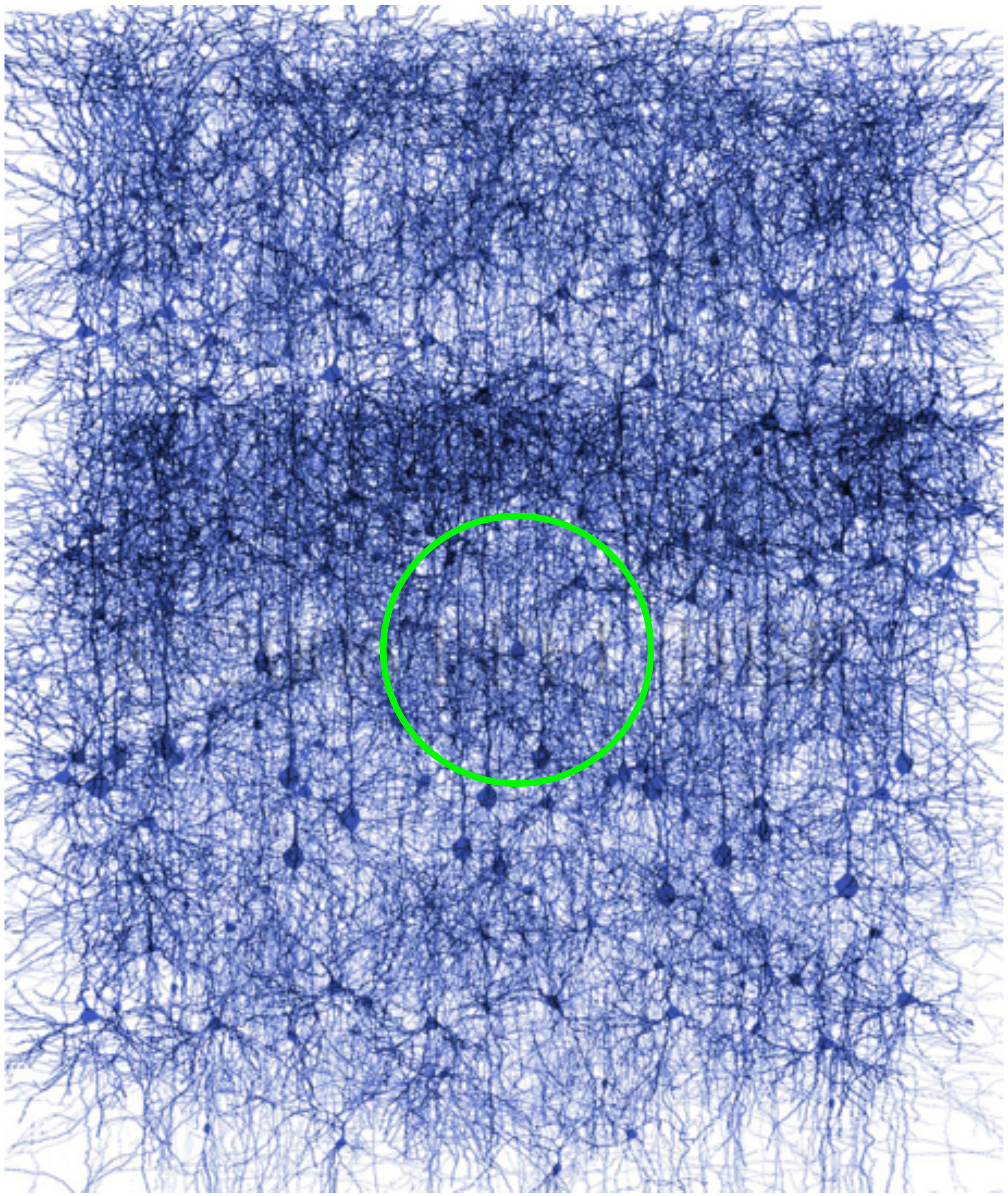


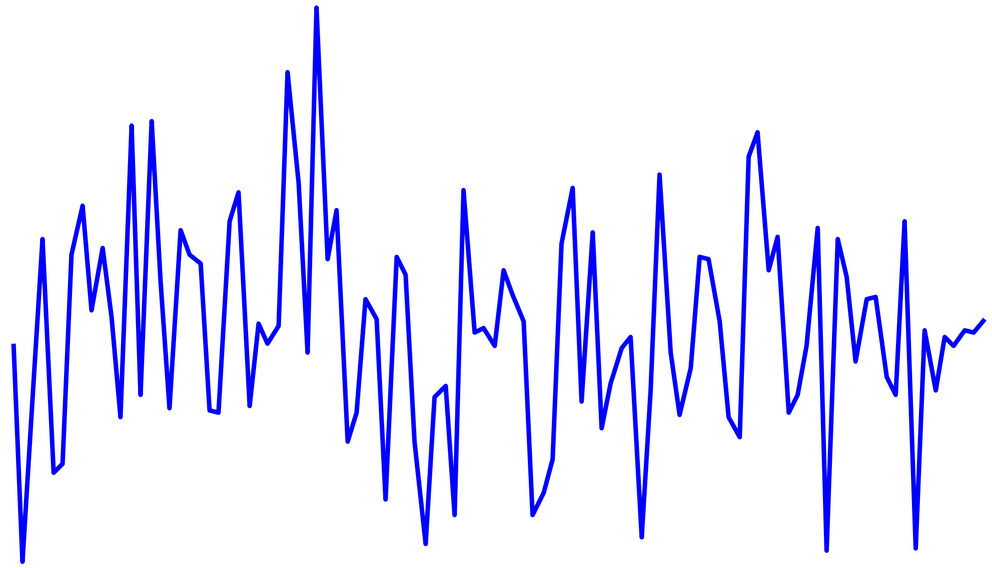
Microscopic → Macroscopic



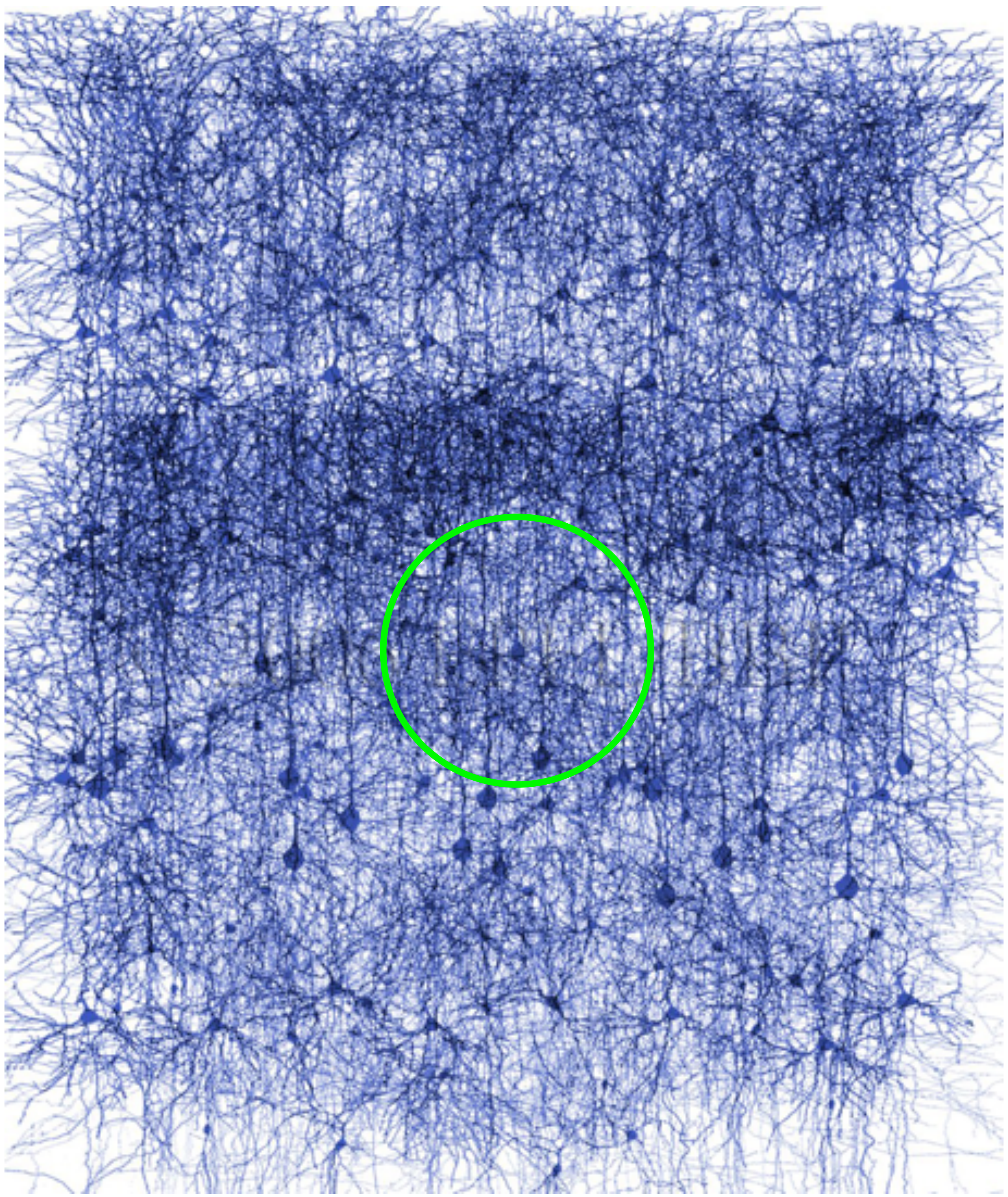


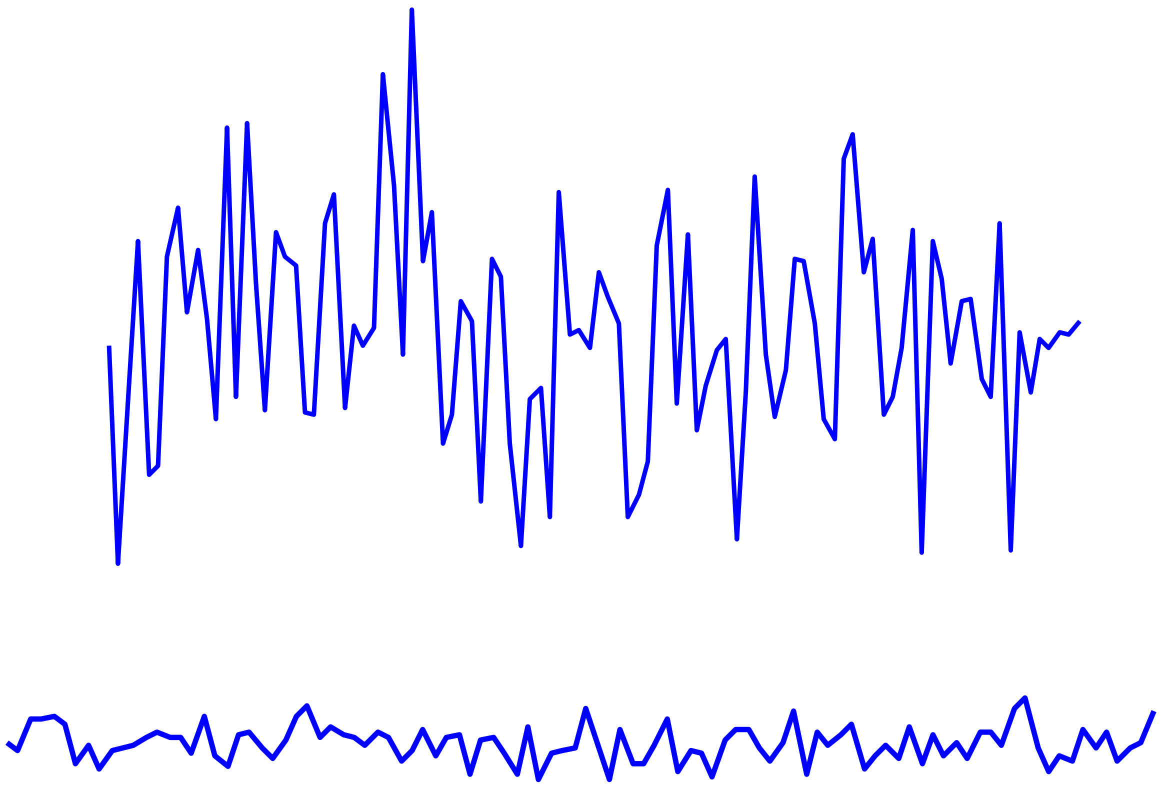
Microscopic → Macroscopic



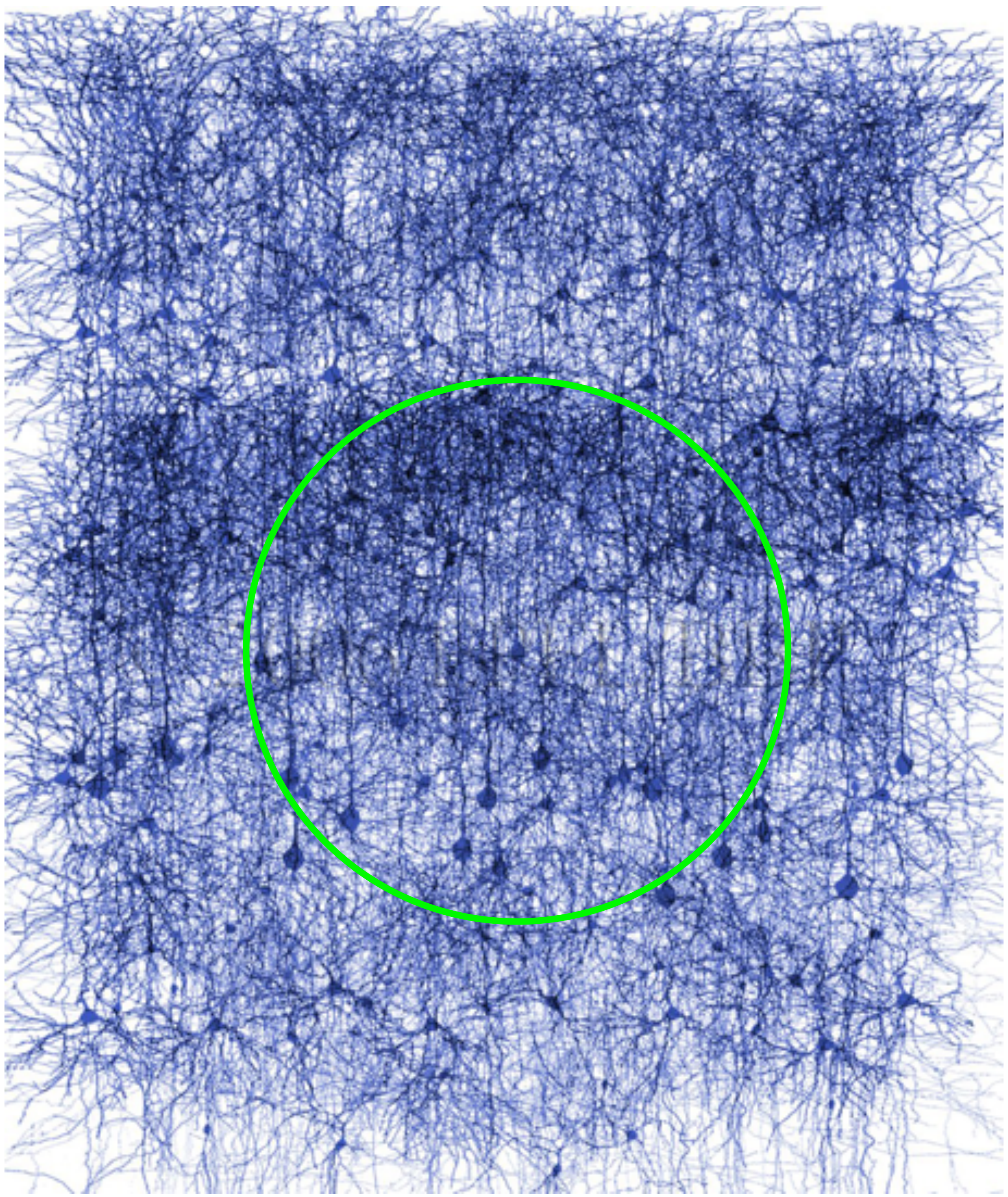


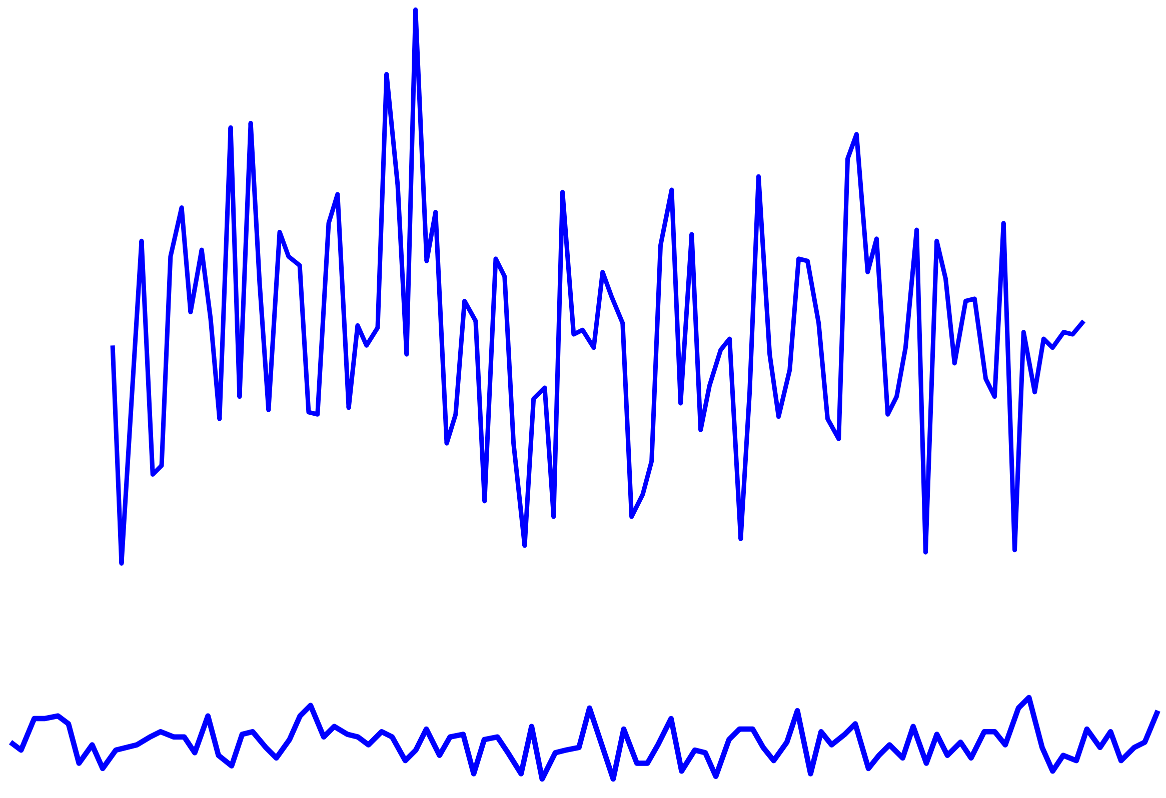
Microscopic → Macroscopic



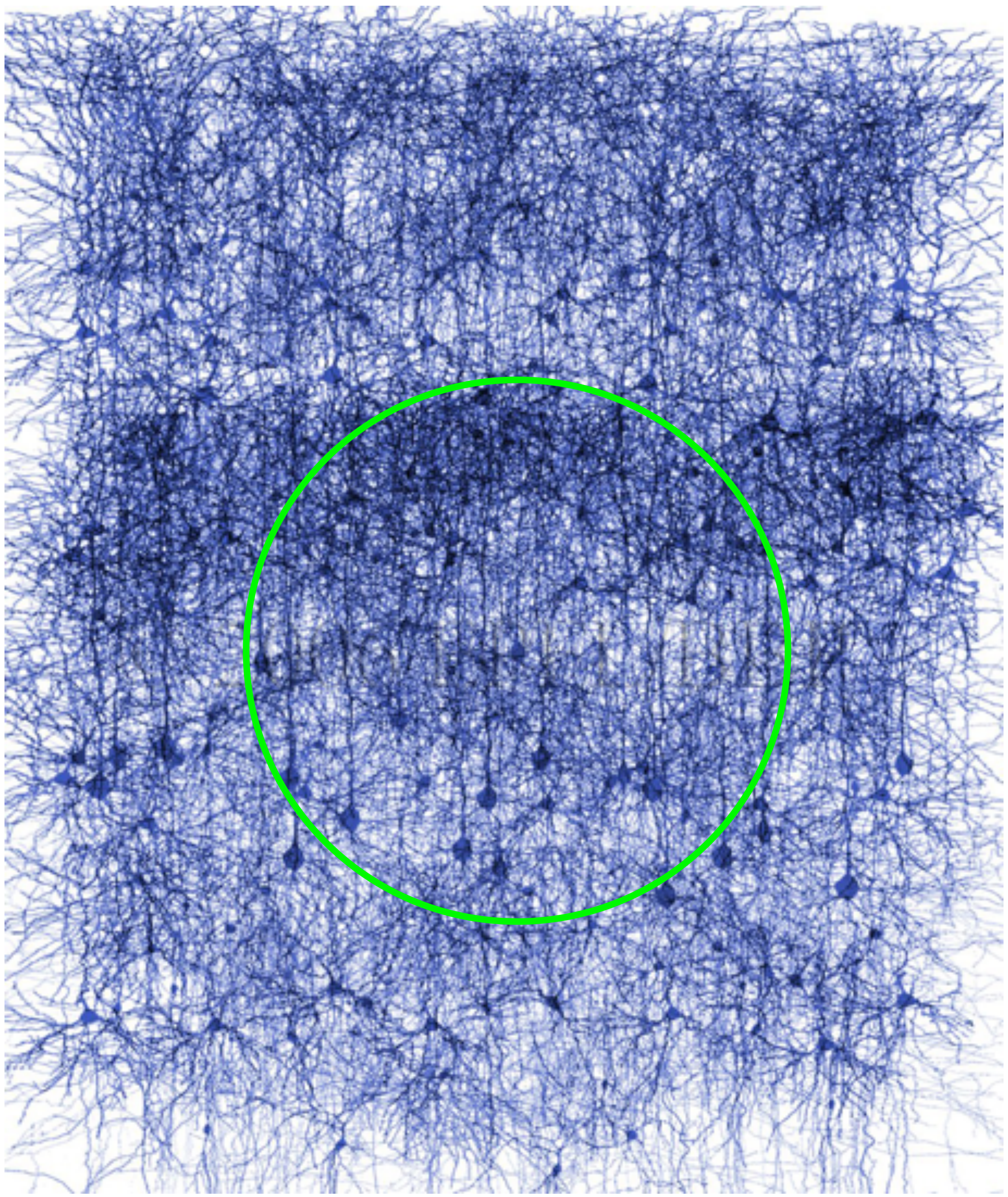


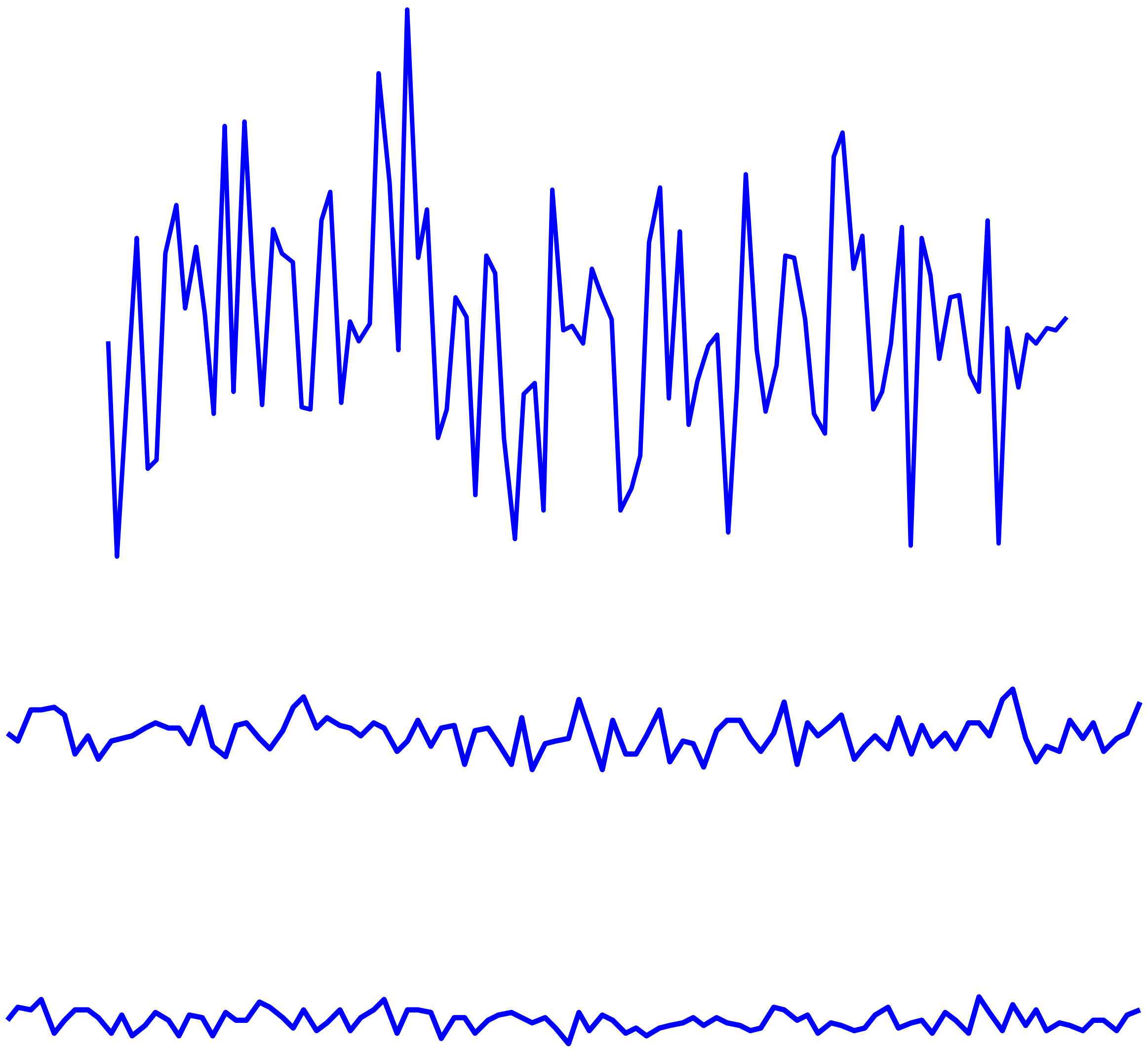
Microscopic → Macroscopic



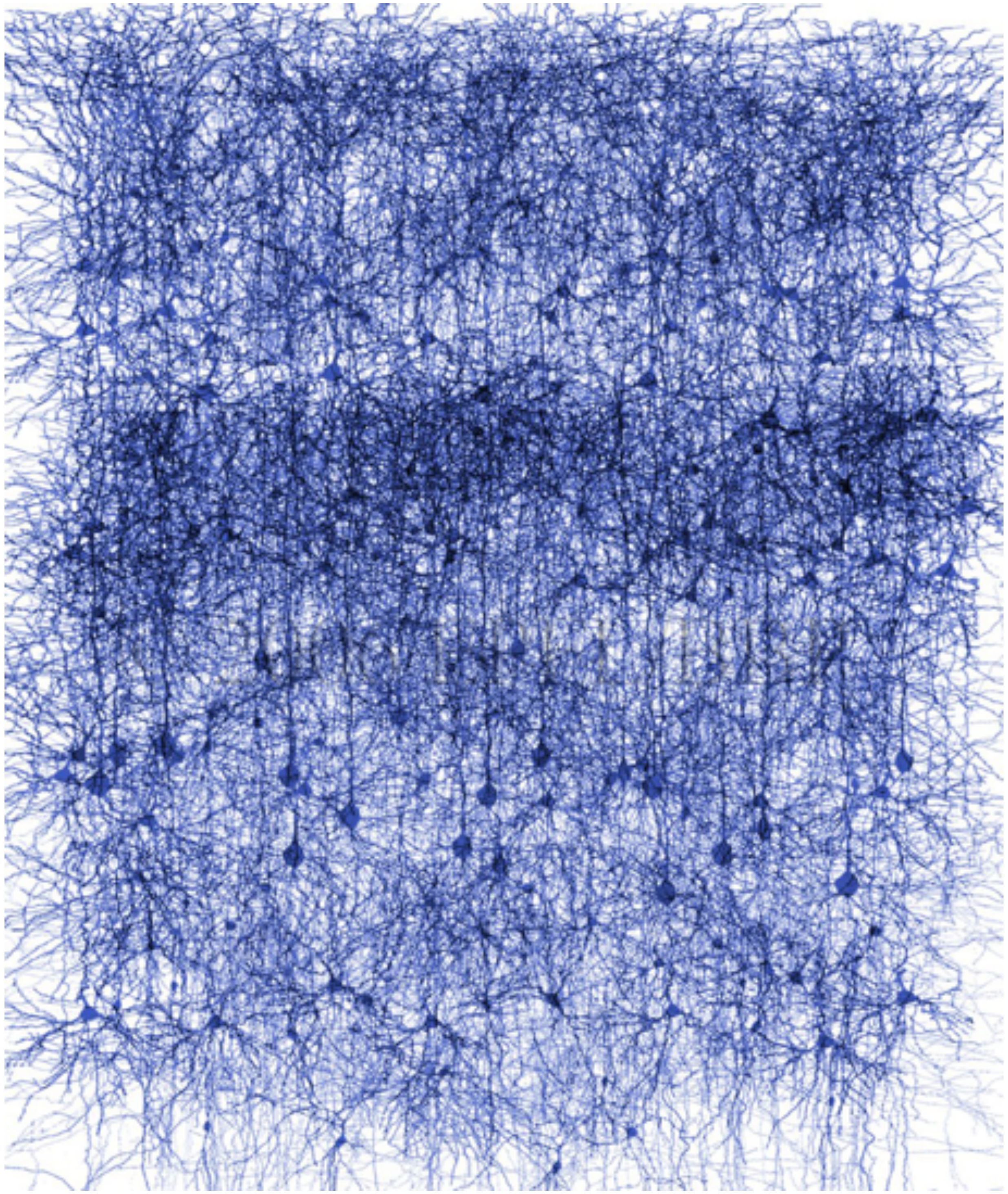


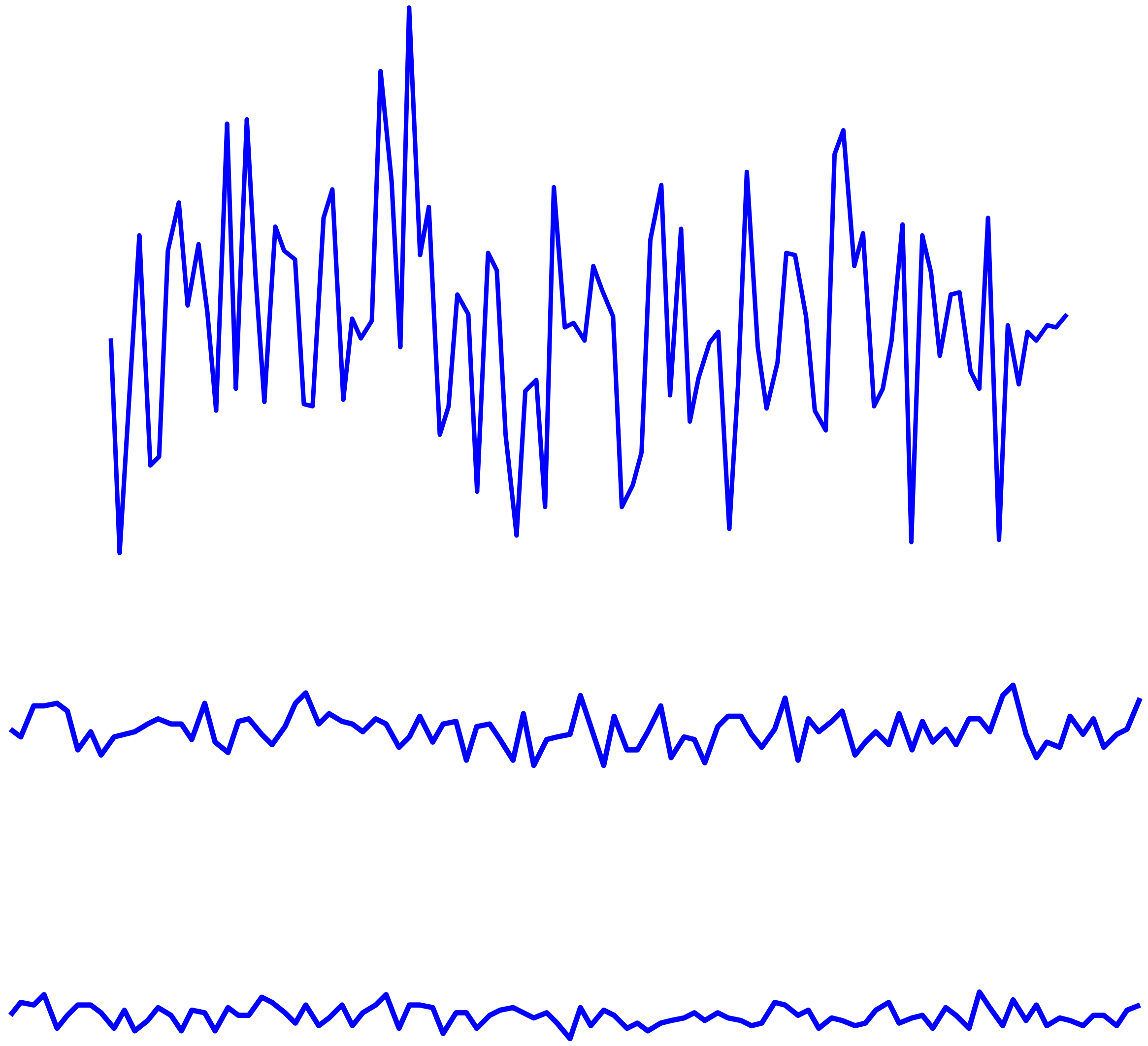
Microscopic → Macroscopic



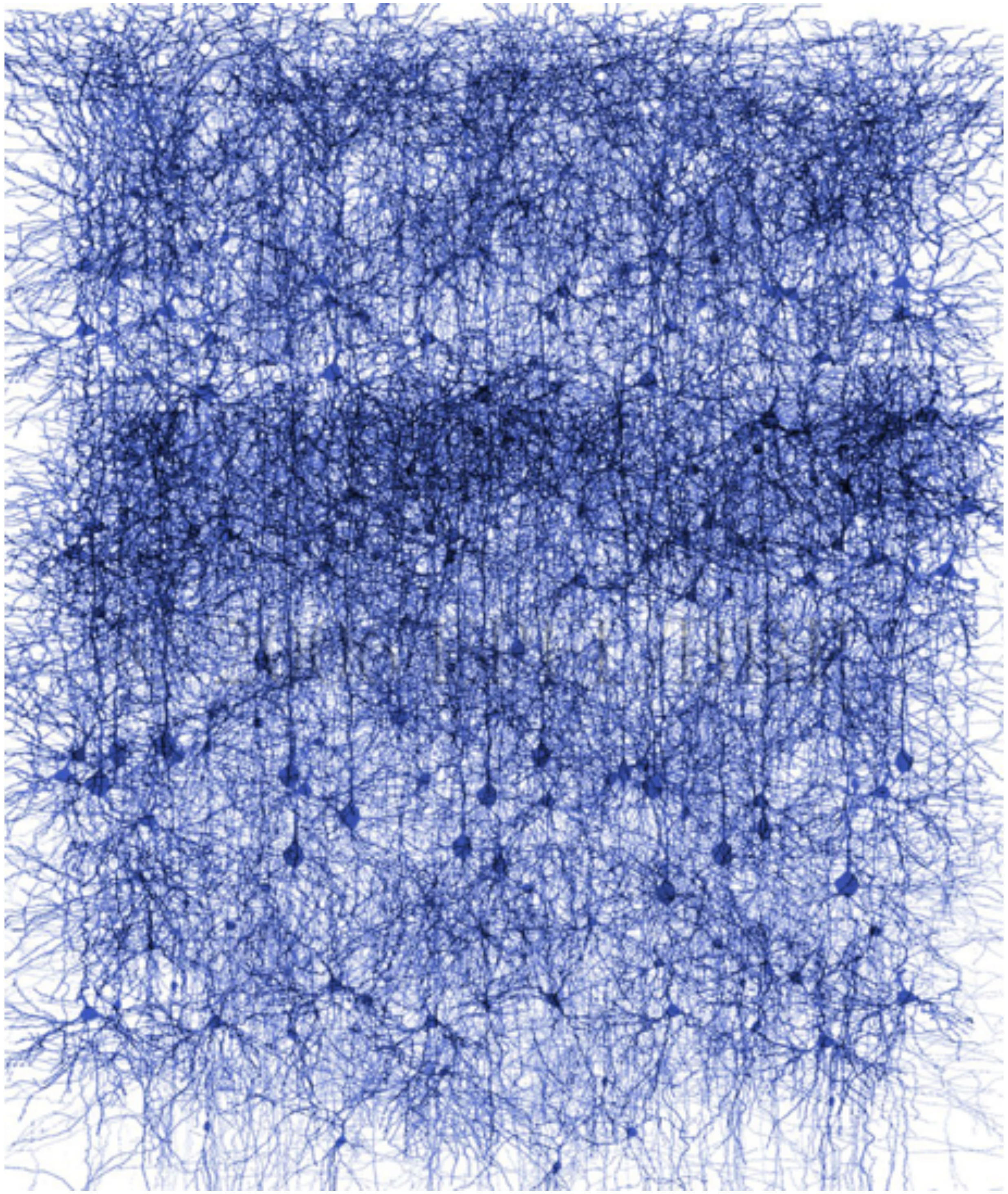


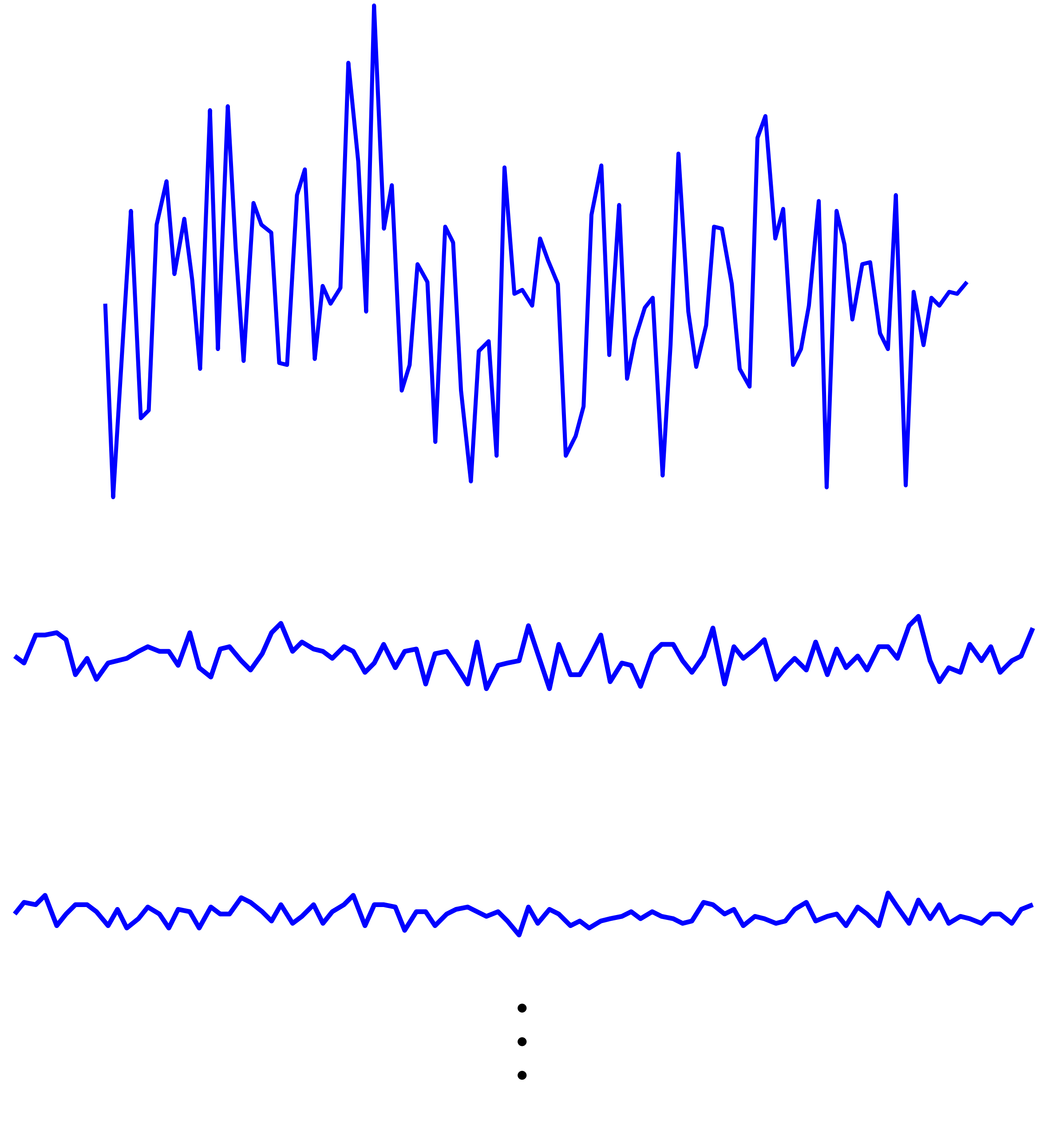
Microscopic → Macroscopic



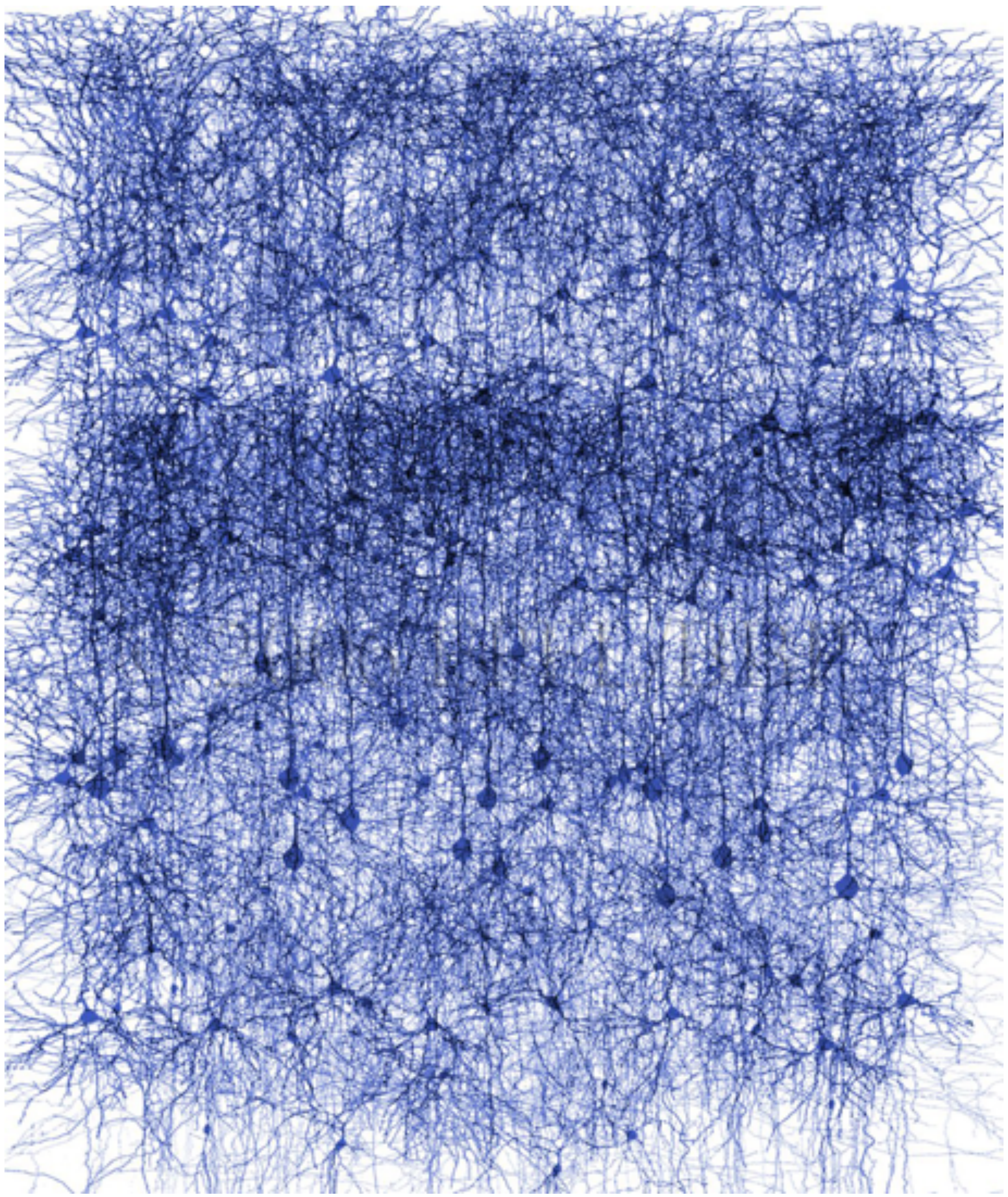


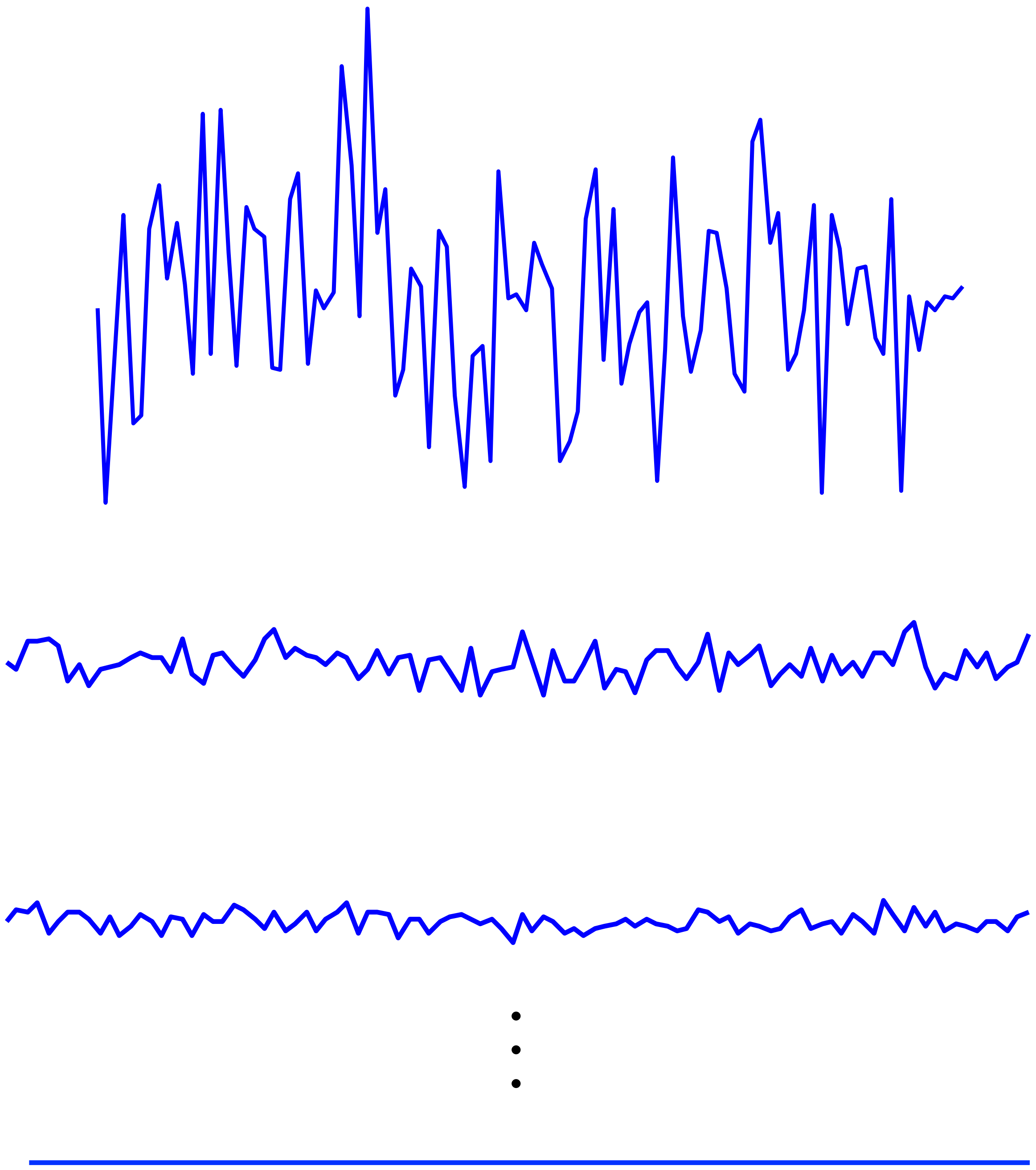
Microscopic → Macroscopic





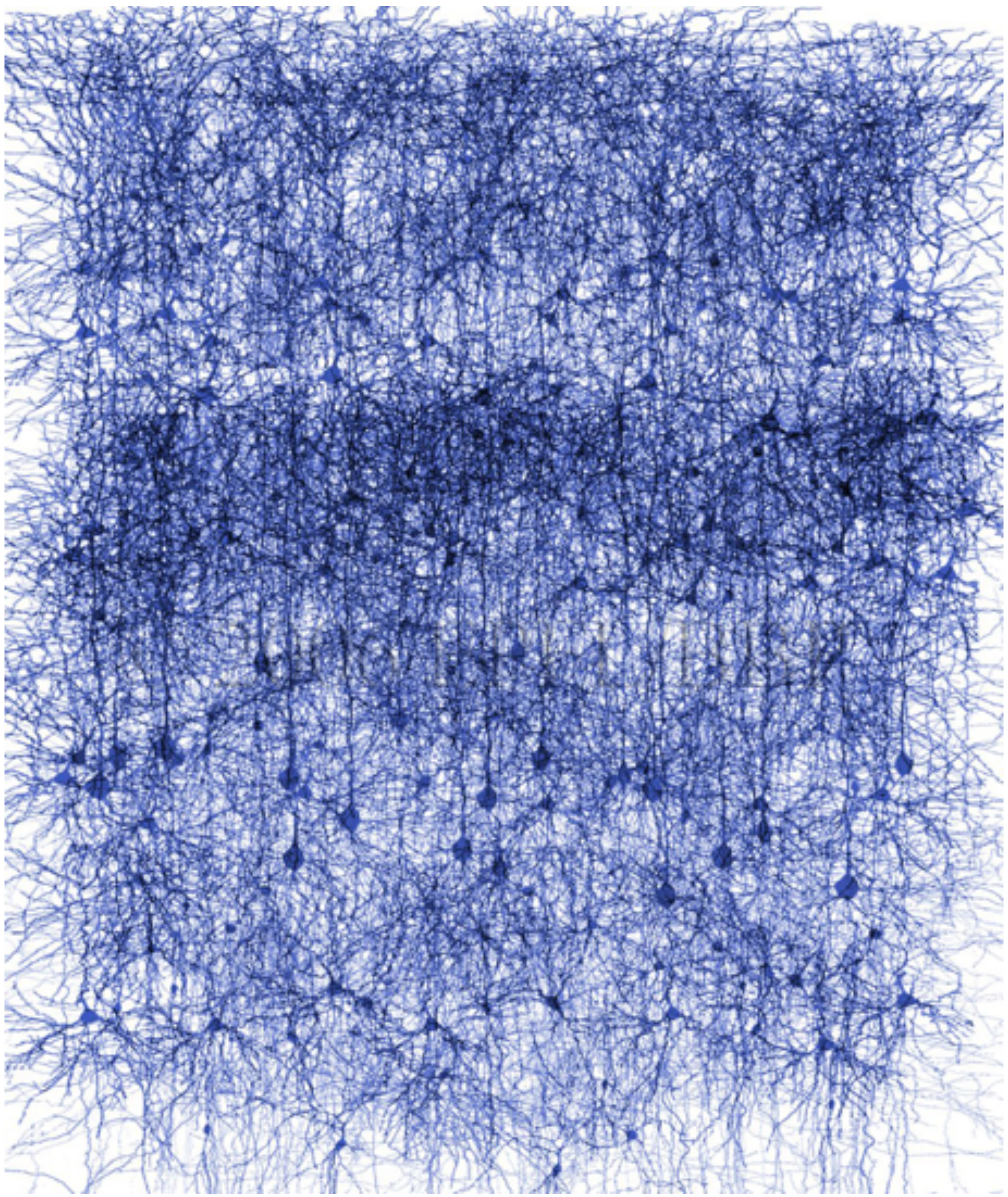
Microscopic → Macroscopic

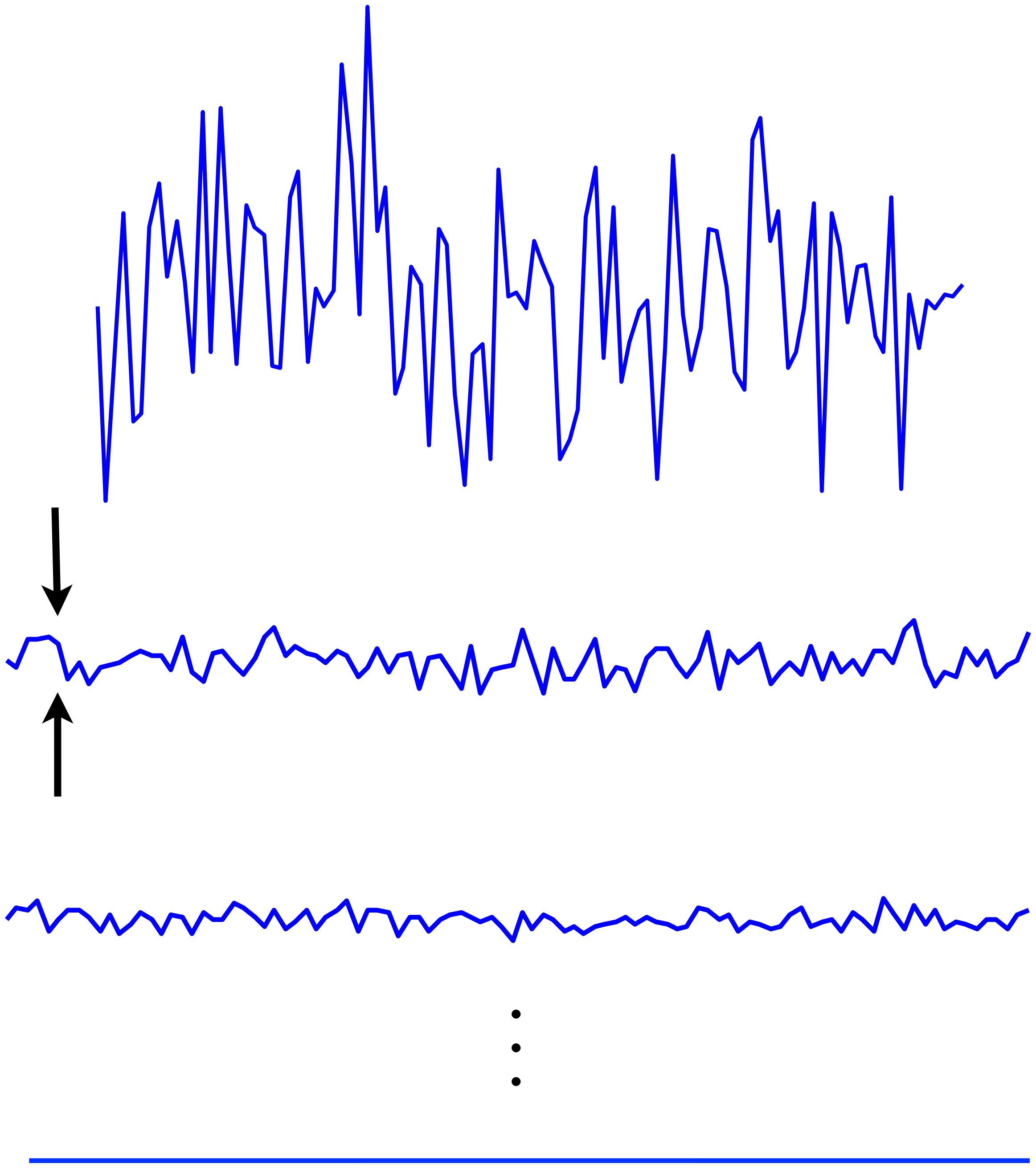




mean field theory

Microscopic → Macroscopic





mean field theory

Microscopic → Macroscopic

variance ∝ N^{-1}

Activity equation
Wilson-Cowan equation

$$\dot{a}_i(t) = -\alpha a_i(t) + f(\sum_j w_{ij} a_j(t) + I_i)$$

Activity equation
Wilson-Cowan equation

$$\dot{a}_i(t) = -\alpha a_i(t) + f(\sum_j w_{ij} a_j(t) + I_i)$$

"activity"

Activity equation
Wilson-Cowan equation

$$\dot{a}_i(t) = -\alpha a_i(t) + f(\sum_j w_{ij} a_j(t) + I_i)$$

Activity equation

Wilson-Cowan equation

rate constant

$$\dot{a}_i(t) = -\alpha a_i(t) + f(\sum_j w_{ij} a_j(t) + I_i)$$

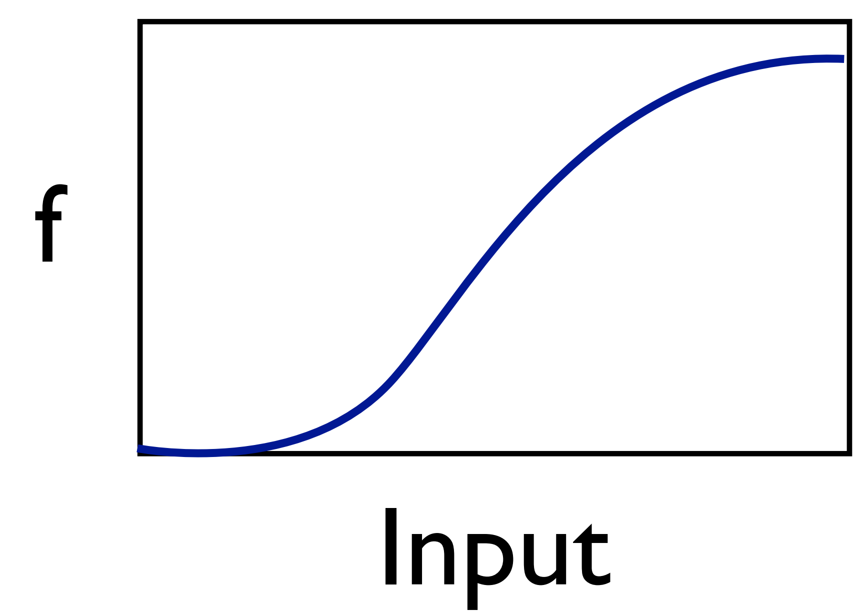
Activity equation
Wilson-Cowan equation

$$\dot{a}_i(t) = -\alpha a_i(t) + f(\sum_j w_{ij} a_j(t) + I_i)$$

Activity equation
Wilson-Cowan equation

$$\dot{a}_i(t) = -\alpha a_i(t) + f(\sum_j w_{ij} a_j(t) + I_i)$$

gain function



Activity equation

Wilson-Cowan equation

$$\dot{a}_i(t) = -\alpha a_i(t) + f(\sum_j w_{ij} a_j(t) + I_i)$$

Activity equation

Wilson-Cowan equation

connection weights

$$\dot{a}_i(t) = -\alpha a_i(t) + f(\sum_j w_{ij} a_j(t) + I_i)$$

Activity equation
Wilson-Cowan equation

$$\dot{a}_i(t) = -\alpha a_i(t) + f(\sum_j w_{ij} a_j(t) + I_i)$$

Activity equation

Wilson-Cowan equation

inputs

$$\dot{a}_i(t) = -\alpha a_i(t) + f(\sum_j w_{ij} a_j(t) + I_i)$$

Activity equation
Wilson-Cowan equation

$$\dot{a}_i(t) = -\alpha a_i(t) + f(\sum_j w_{ij} a_j(t) + I_i)$$

Activity equation

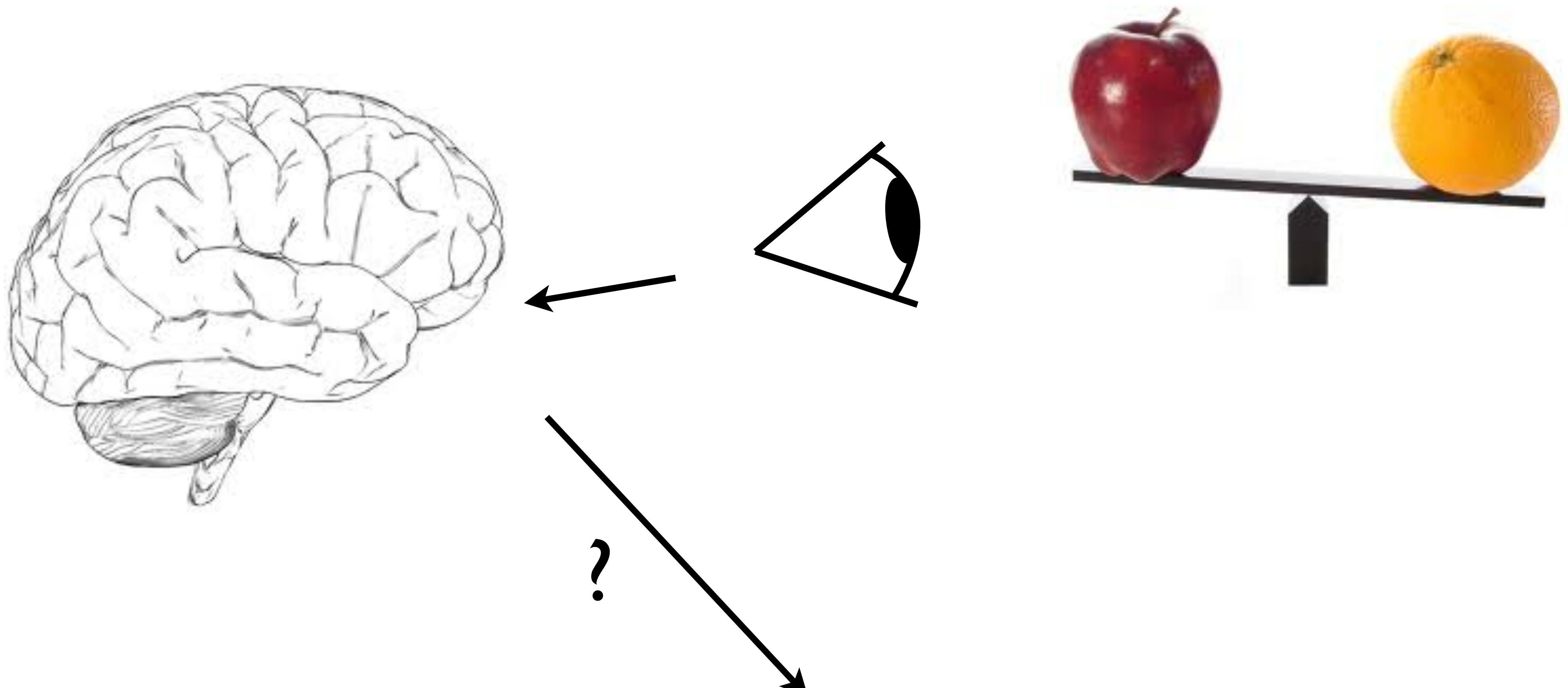
Wilson-Cowan equation

$$\dot{a}_i(t) = -\alpha a_i(t) + f(\sum_j w_{ij} a_j(t) + I_i)$$

purely phenomenological

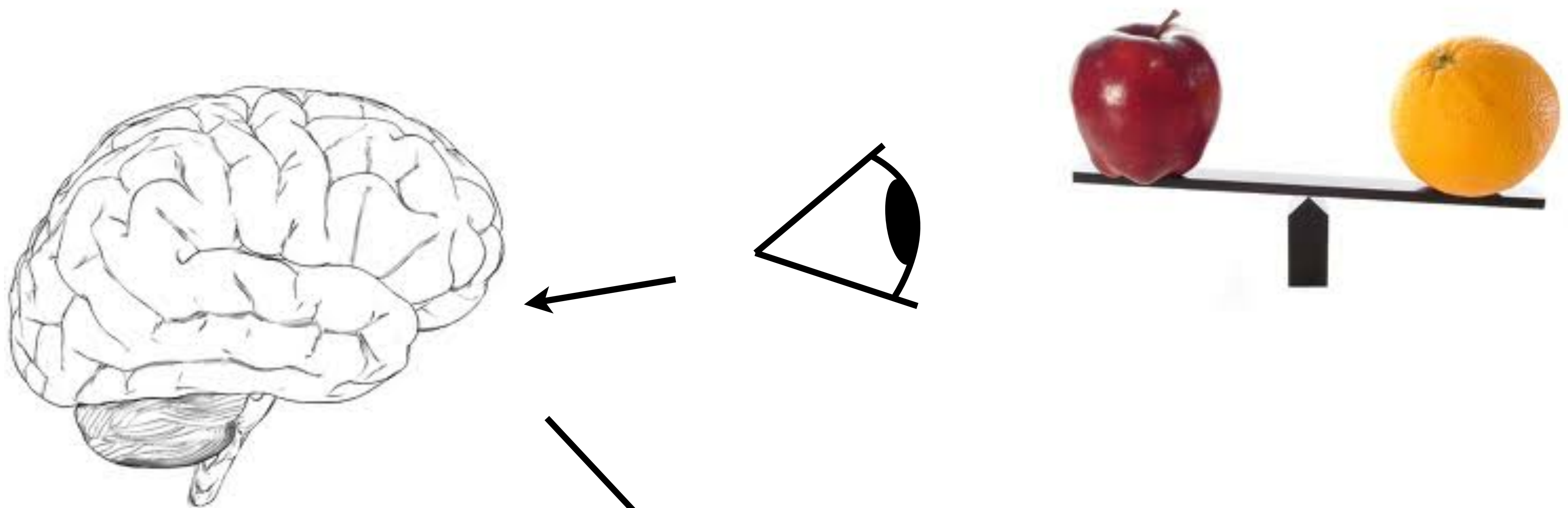
Want to derive from neurons

Brain as a map from inputs I to outputs a



$$\dot{a}_i(t) = -\alpha a_i(t) + f(\sum_j w_{ij} a_j(t) + I_i)$$

Brain as a map from inputs I to outputs a

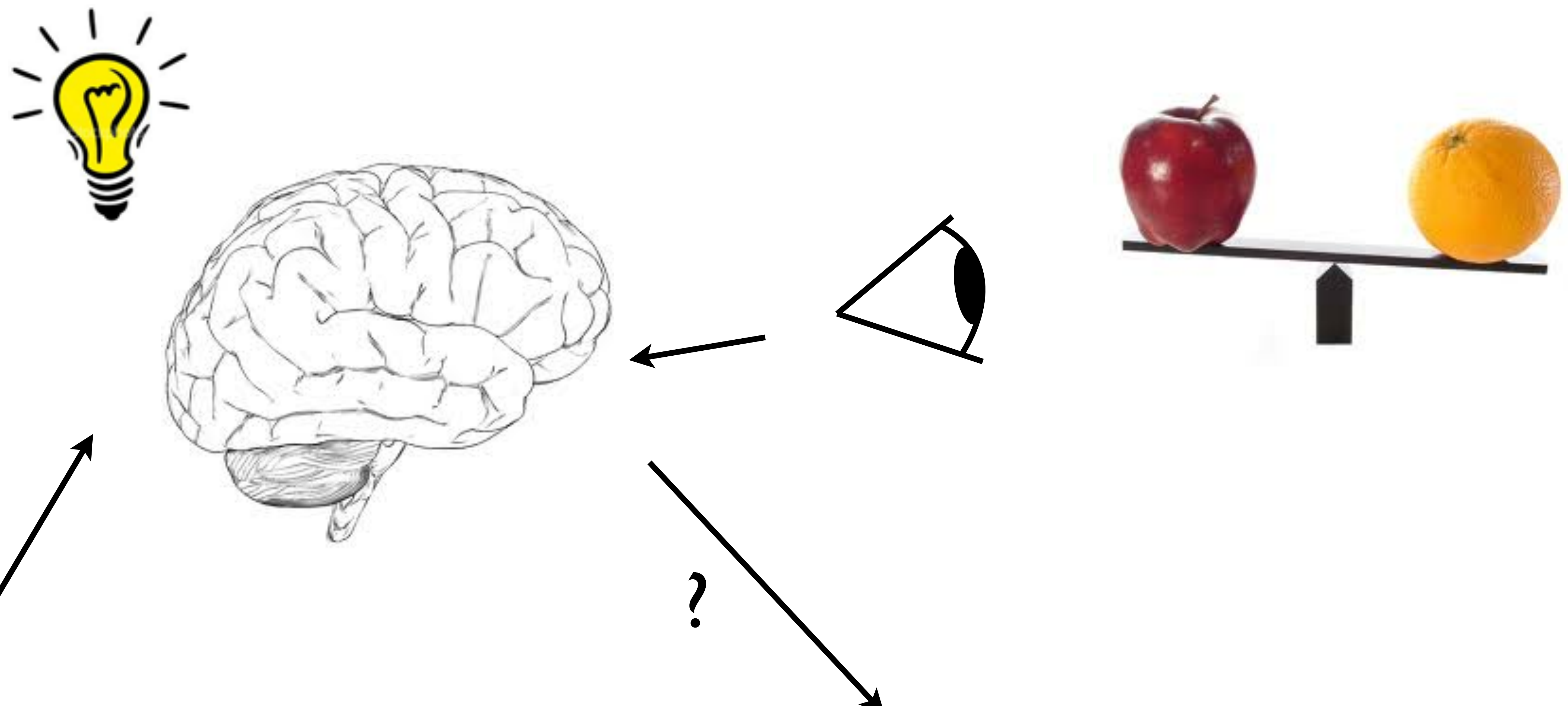


?

$$\dot{a}_i(t) = -\alpha a_i(t) + f(\sum_j w_{ij} a_j(t) + I_i)$$

Learning

Brain as a map from inputs I to outputs a



$$\dot{a}_i(t) = -\alpha a_i(t) + f(\sum_j w_{ij} a_j(t) + I_i)$$

Learning

Example learning rules

$$\tau \dot{w}_{ij} = a_i a_j - w_{ij}$$ Hebbian rule

Example learning rules

$$\tau \dot{w}_{ij} = a_i a_j - w_{ij}$$ Hebbian rule

$$\tau \dot{w}_{ij} = C_{ij} - w_{ij}$$ Correlation rule

Example learning rules

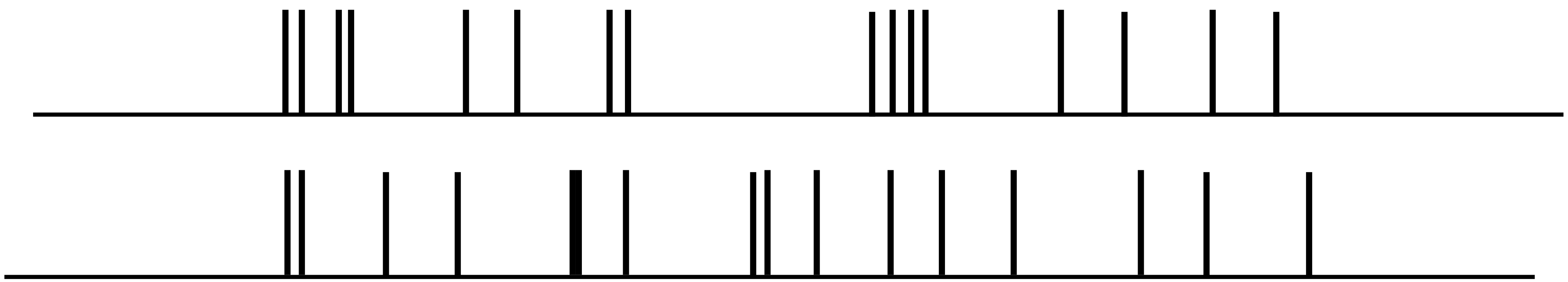
$$\tau \dot{w}_{ij} = a_i a_j - w_{ij}$$ Hebbian rule

$$\tau \dot{w}_{ij} = C_{ij} - w_{ij}$$ Correlation rule

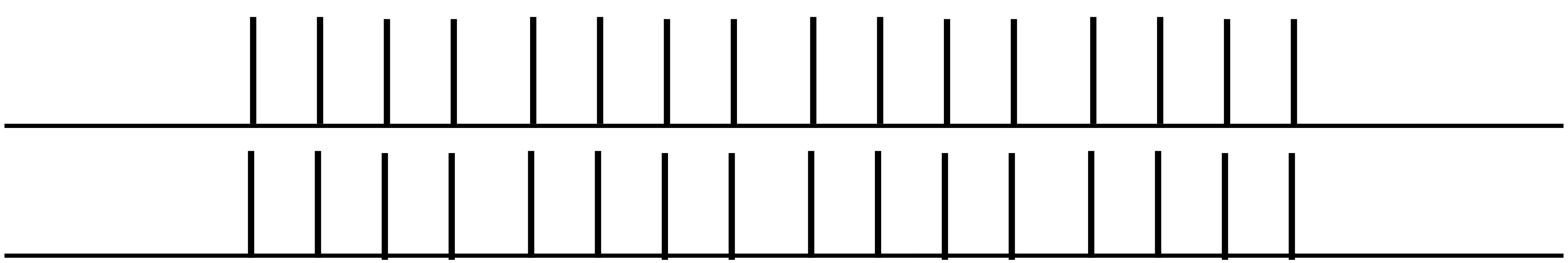
but activity equations ignore correlations

Correlations

Poisson



Synchronized



time

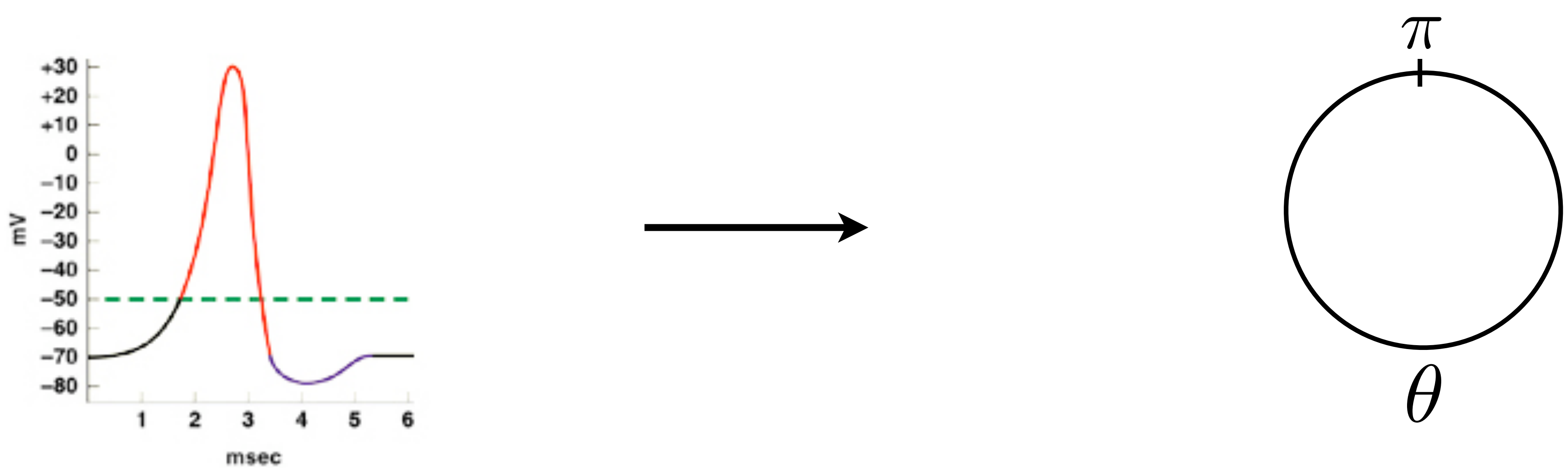
“Generalized” activity equations

$$\dot{a}_i(t) = -\alpha a_i(t) + f(\sum_j w_{ij} a_j(t) + I_i) + G[C_{ij}]$$

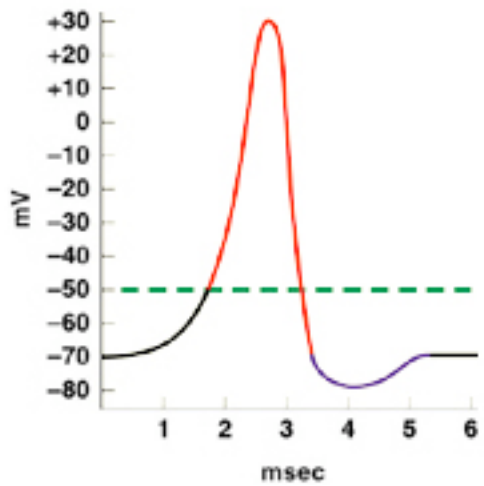
$$\dot{C}_{ij}(t) = \psi[C_{ij}, a_i, a_j]$$

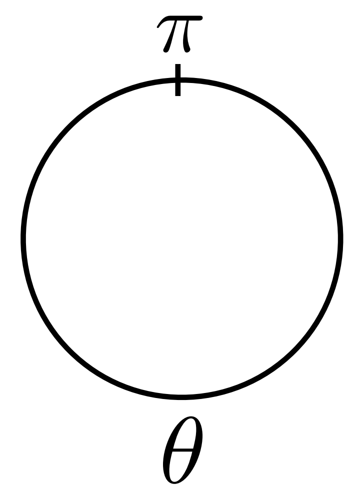
Compute C_{ij} from neurons

Neuron phase models



Neuron phase models



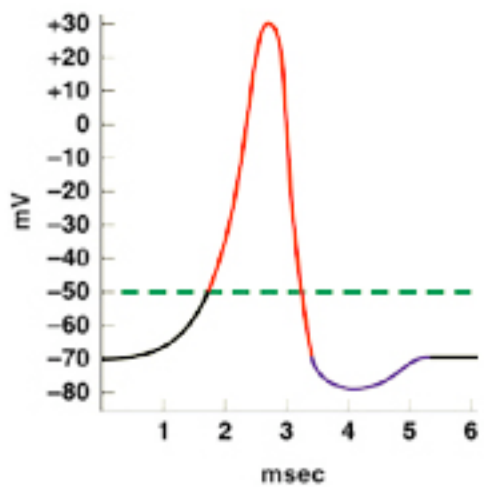


$$\frac{dv}{dt} = I + v^2$$

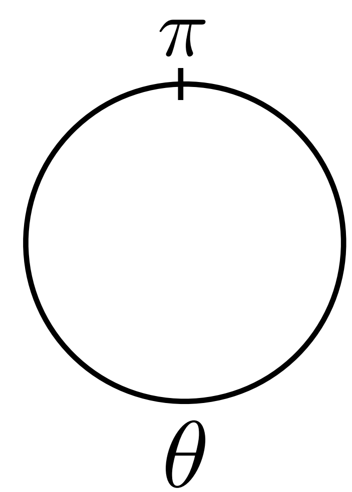
$$v = \tan(\theta/2)$$

$$\frac{d\theta}{dt} = 1 - \cos\theta + I(1 + \cos\theta)$$

Neuron phase models



$\longrightarrow$



$$\frac{dv}{dt} = I + v^2$$

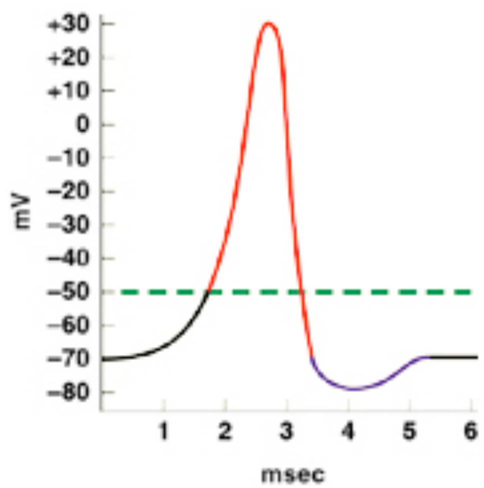
$$\longrightarrow$$

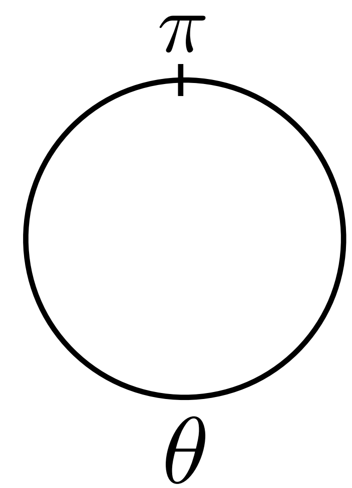
$$v = \tan(\theta/2)$$

$$\frac{d\theta}{dt} = 1 - \cos\theta + I(1 + \cos\theta)$$

Quadratic integrate-and-fire

Neuron phase models





$$\frac{dv}{dt} = I + v^2$$

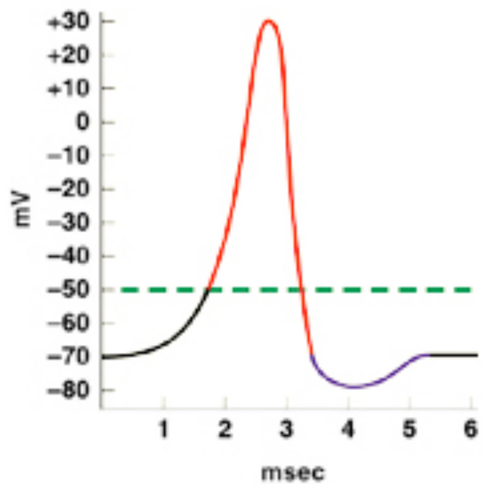
$$v = \tan(\theta/2)$$

$$\frac{d\theta}{dt} = 1 - \cos\theta + I(1 + \cos\theta)$$

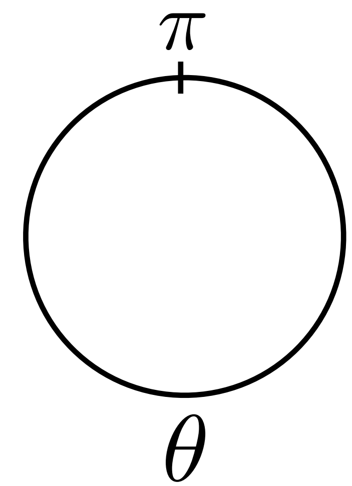
Quadratic integrate-and-fire

Theta model

Neuron phase models



→



$$\frac{dv}{dt} = I + v^2$$

→

$$v = \tan(\theta/2)$$

$$\frac{d\theta}{dt} = 1 - \cos\theta + I(1 + \cos\theta)$$

Quadratic integrate-and-fire

Theta model

Simple phase model

$$\frac{d\theta}{dt} = I$$

Neuron model with coupling

$$\dot{\theta}_i = f_i(\theta) + \alpha_i u(t)$$

$$\dot{u}_i + \beta u_i = \frac{\beta}{N} \sum_j w_{ij} \delta(t - t_j^s)$$

Neuron model with coupling

$$\dot{\theta}_i = f_i(\theta) + \alpha_i u(t)$$

$$\dot{u}_i + \beta u_i = \frac{\beta}{N} \sum_j w_{ij} \delta(t - t_j^s)$$

spike times of neuron *j*

Neuron model with coupling

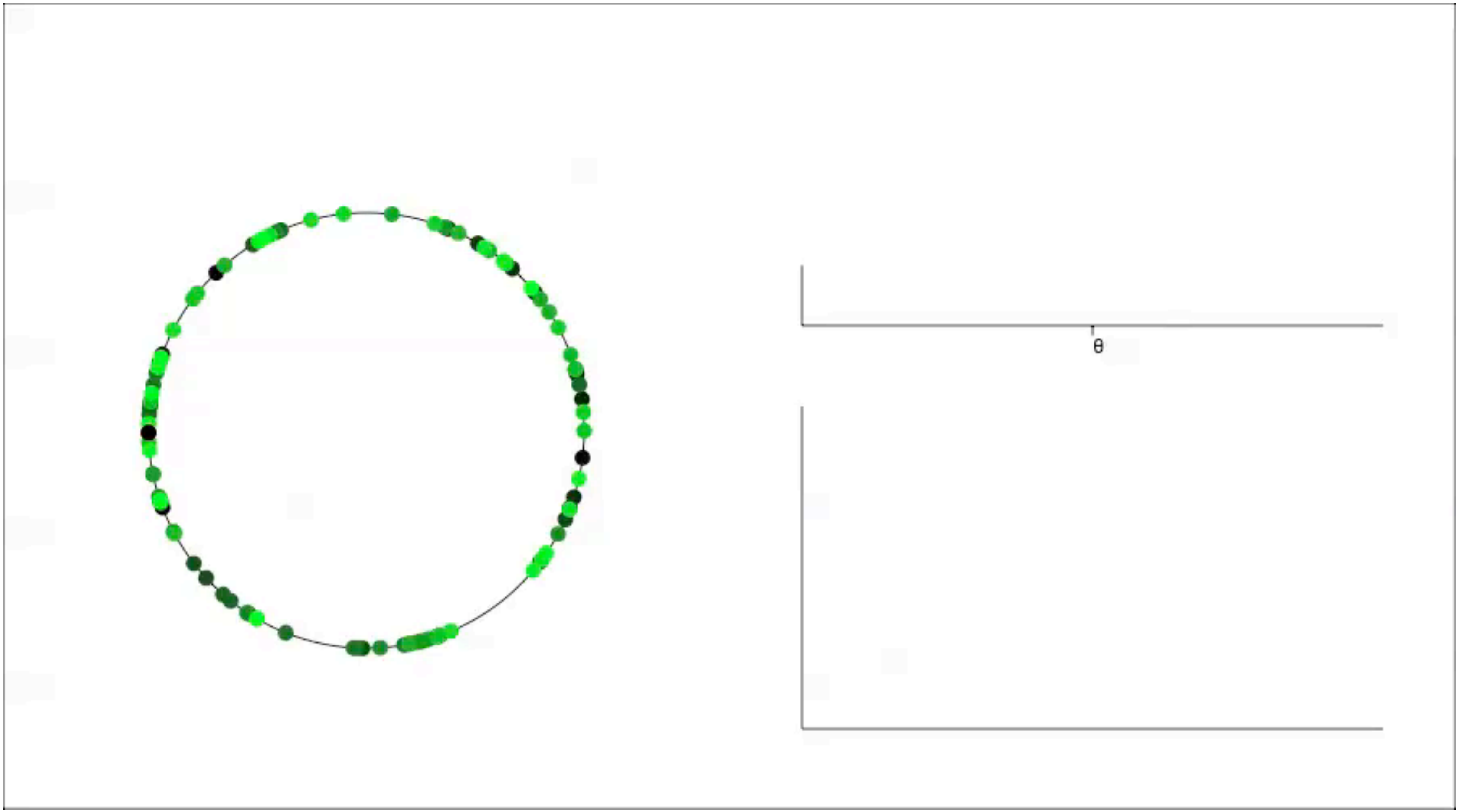
$$\dot{\theta}_i = f_i(\theta) + \alpha_i u(t)$$

$$\dot{u}_i + \beta u_i = \frac{\beta}{N} \sum_j w_{ij} \delta(t - t_j^s)$$

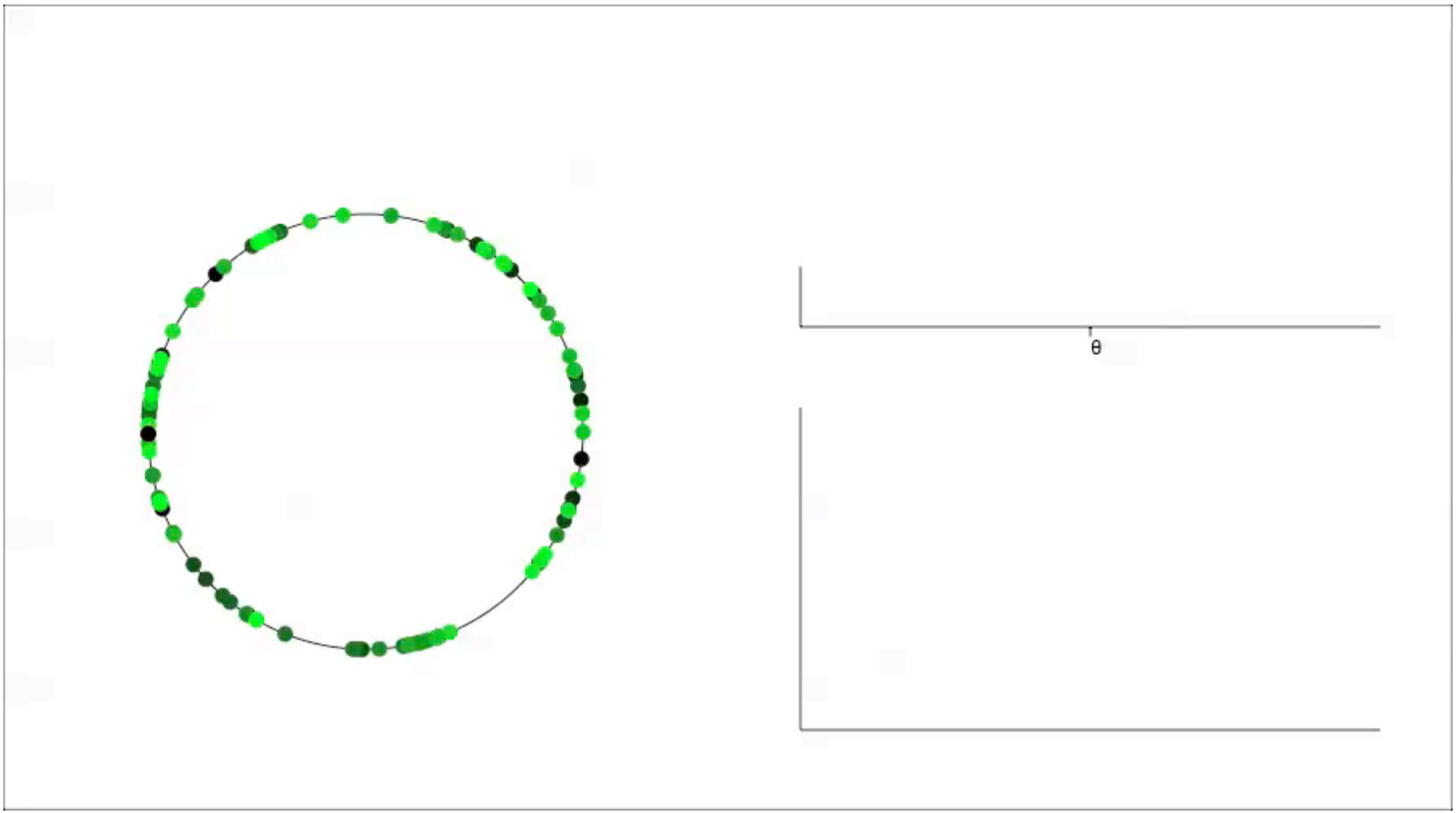
spike times of neuron *j*

Global coupling: w_{ij} = const

Correlations from finite size effects

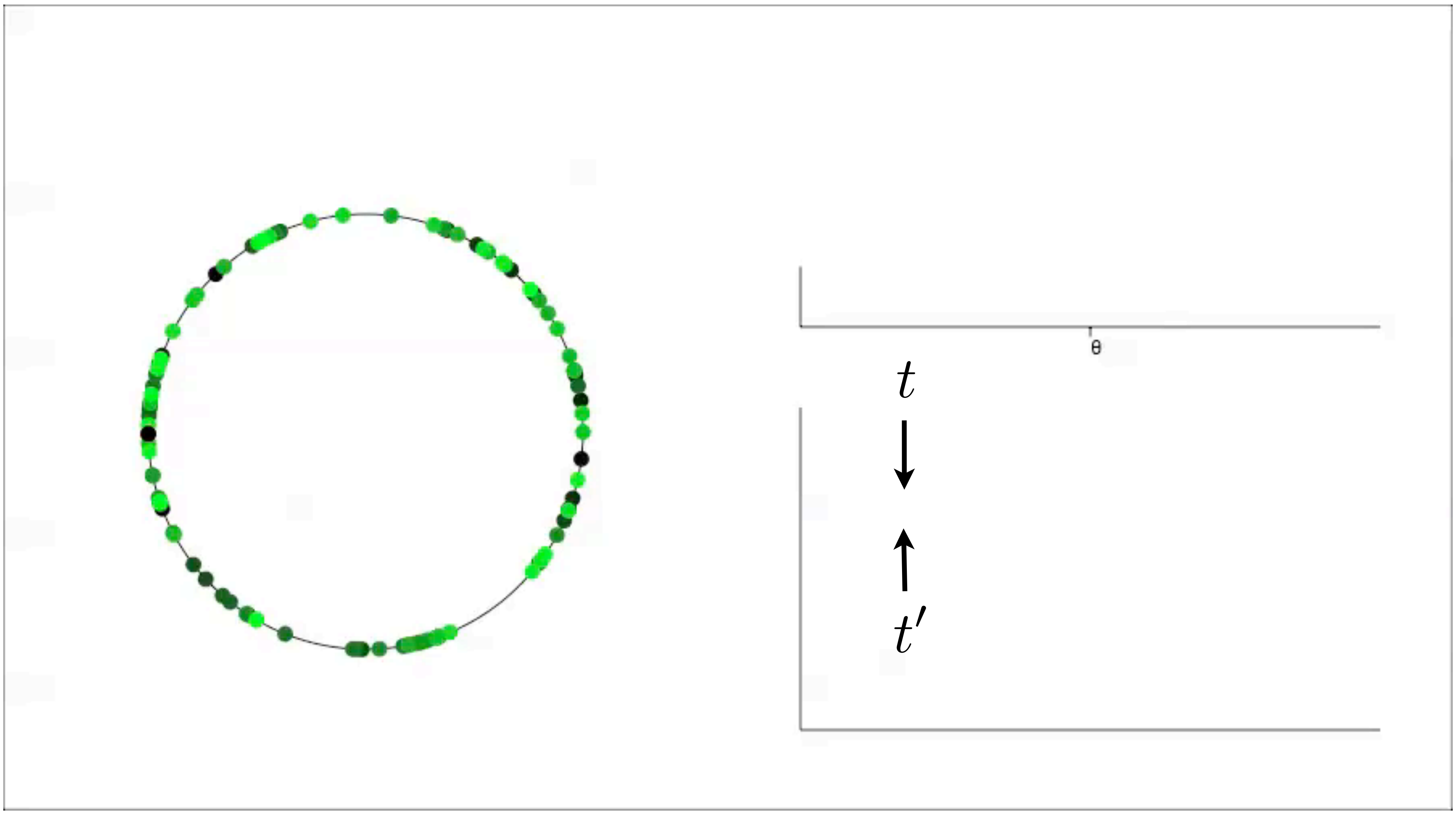


Correlations from finite size effects



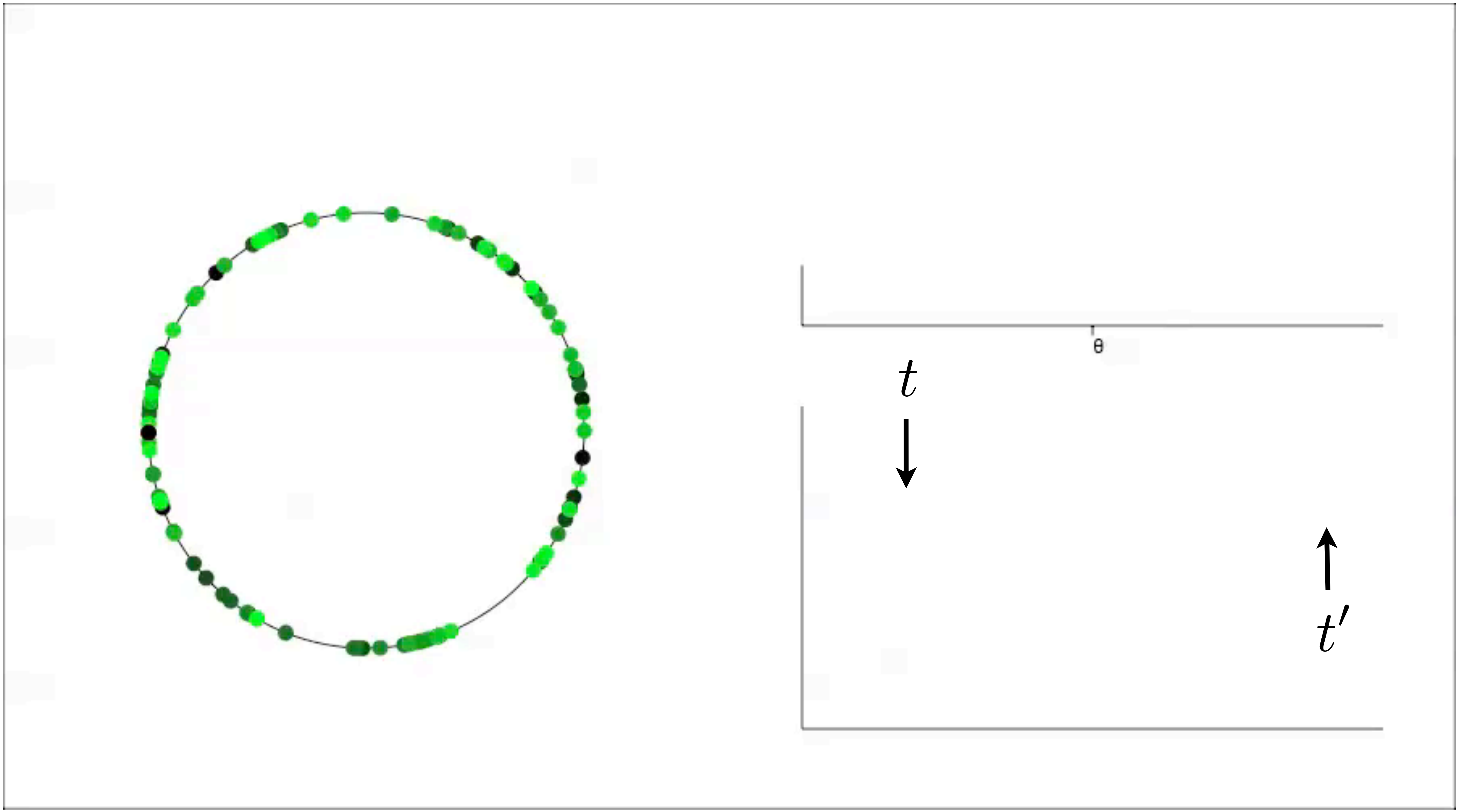
$$C(t,t') = \langle (u(t)-\bar{u})(u(t')-\bar{u}) \rangle$$

Correlations from finite size effects



$$C(t,t') = \langle (u(t) - \bar{u})(u(t') - \bar{u}) \rangle$$

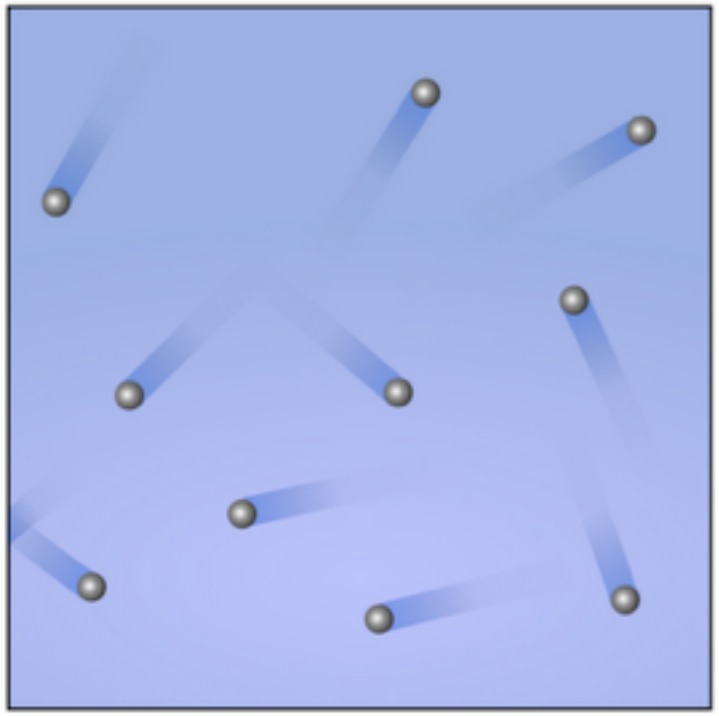
Correlations from finite size effects



$$C(t,t') = \langle (u(t) - \bar{u})(u(t') - \bar{u}) \rangle$$

Kinetic theory

↔



Joule

Boltzmann

Kinetic theory

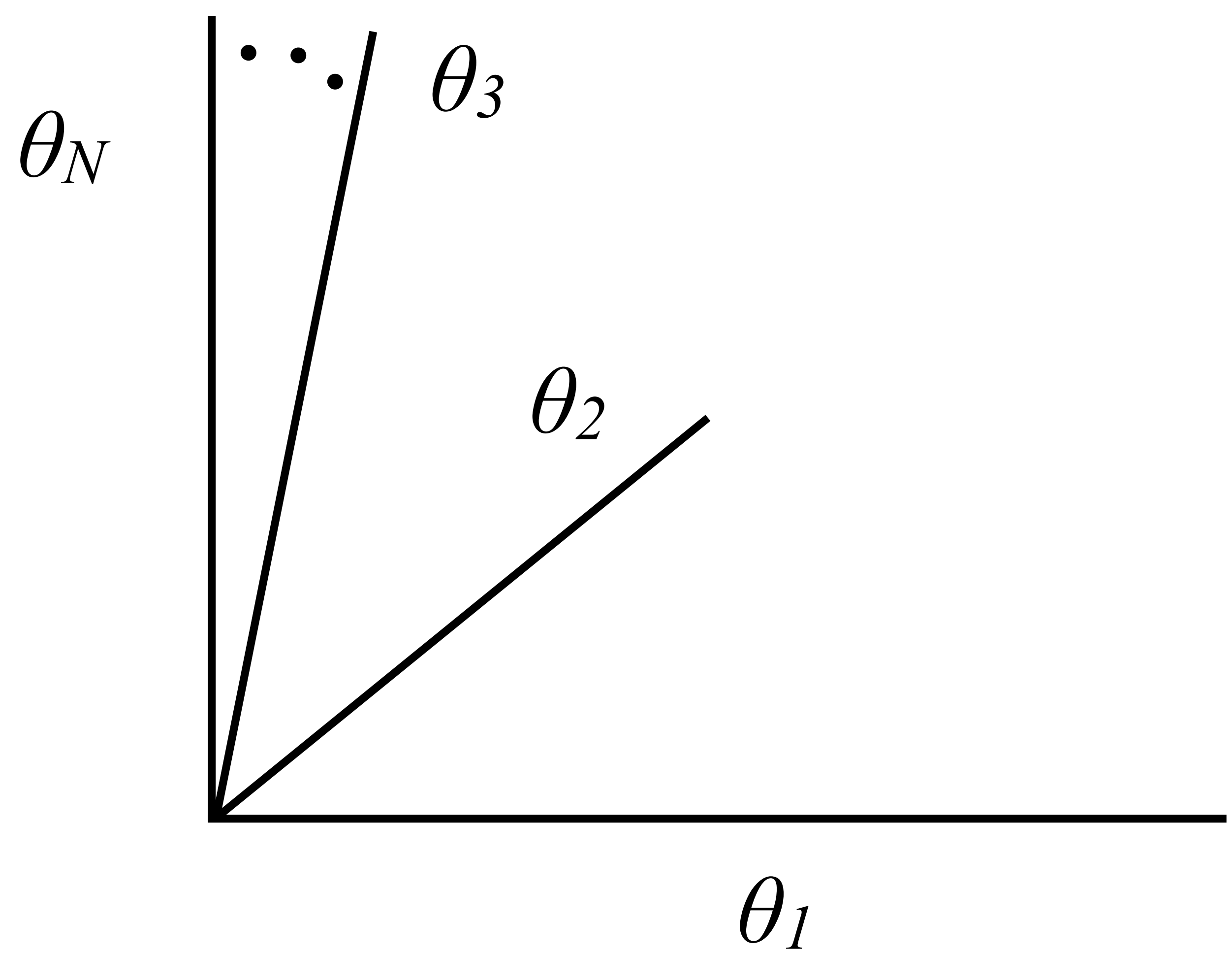
Derive macroscopic equations
from microscopic dynamics

Kinetic theory

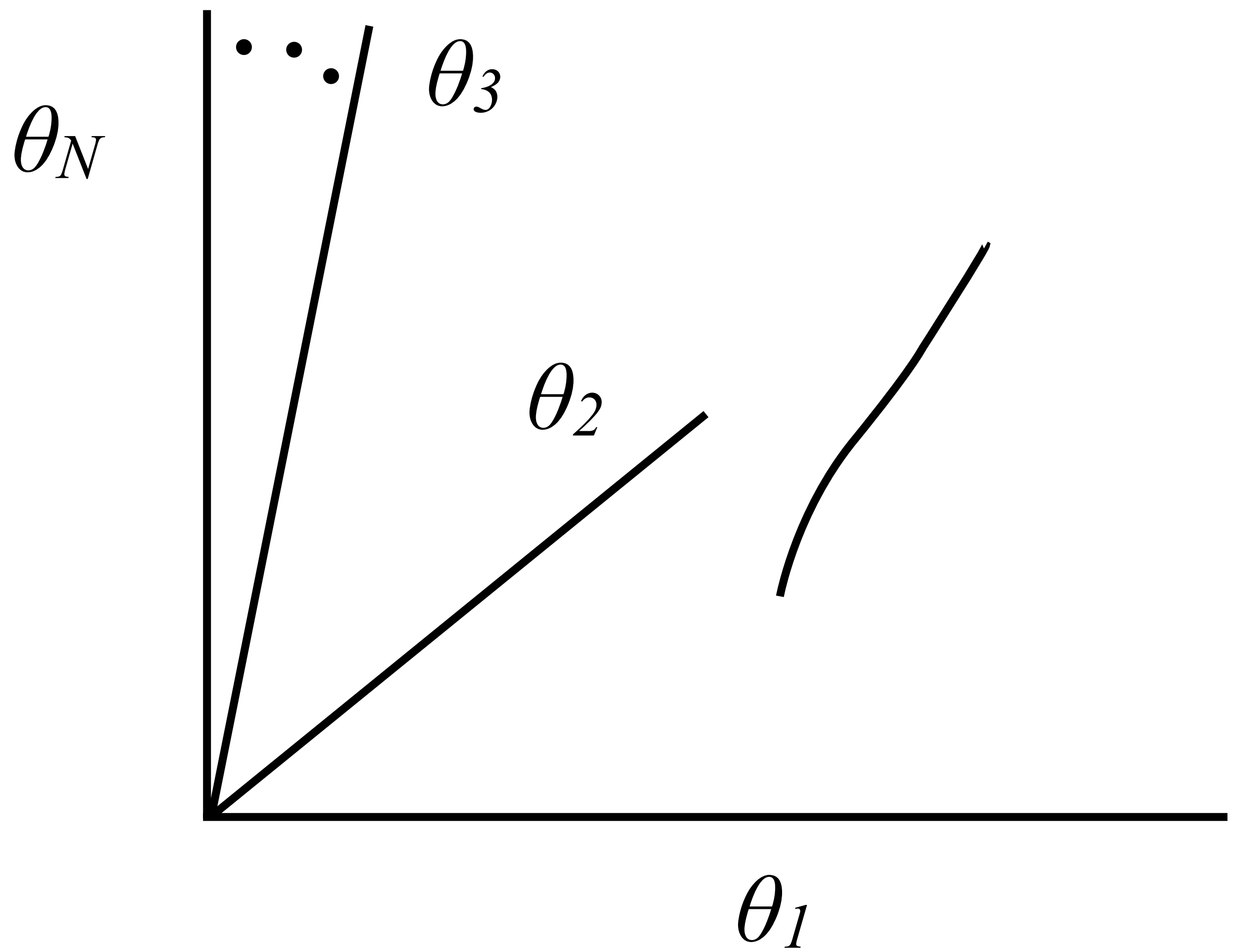
Derive macroscopic equations
from microscopic dynamics

microscopic → probabilistic → activity

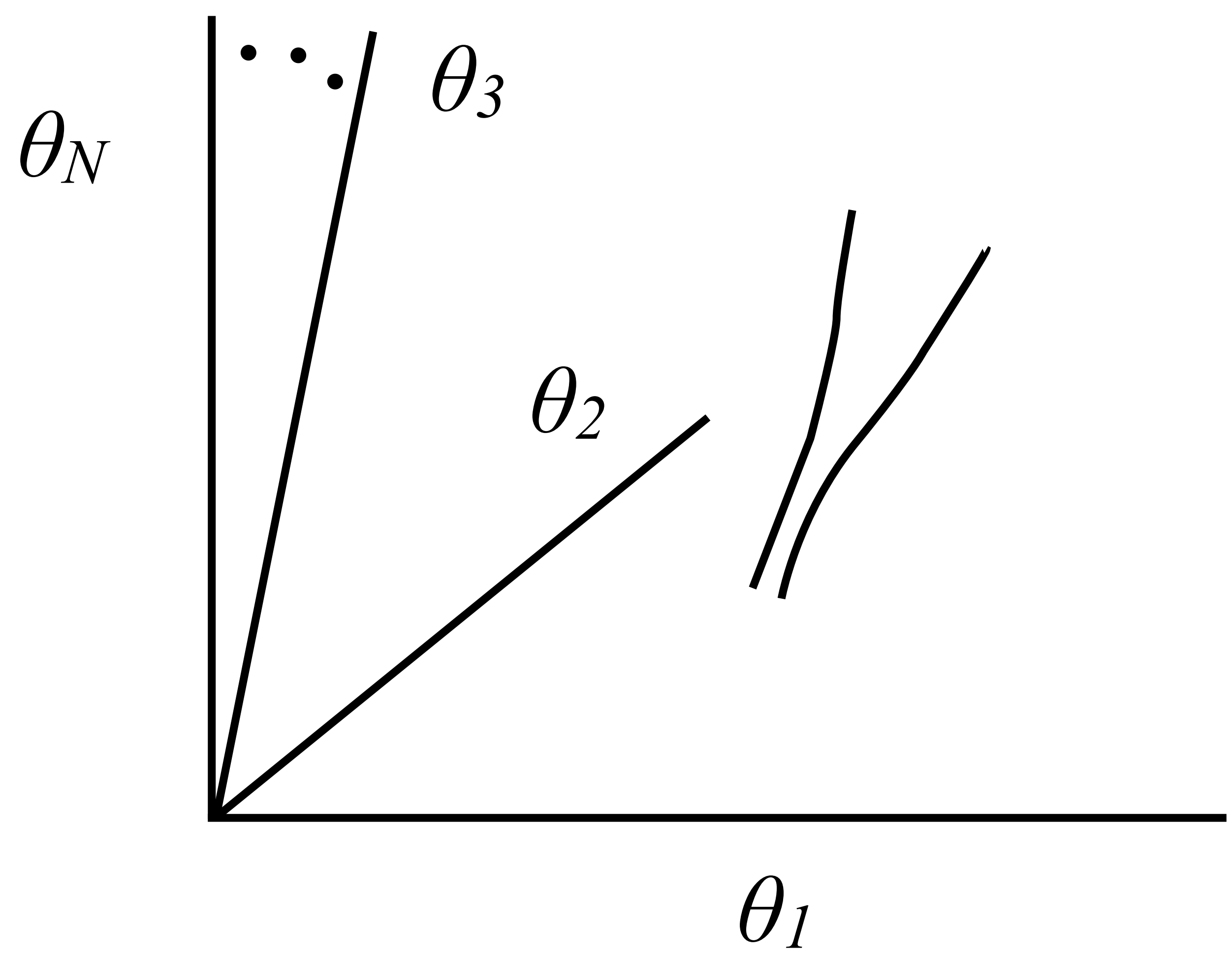
Probability density evolution in N dimensions



Probability density evolution in N dimensions



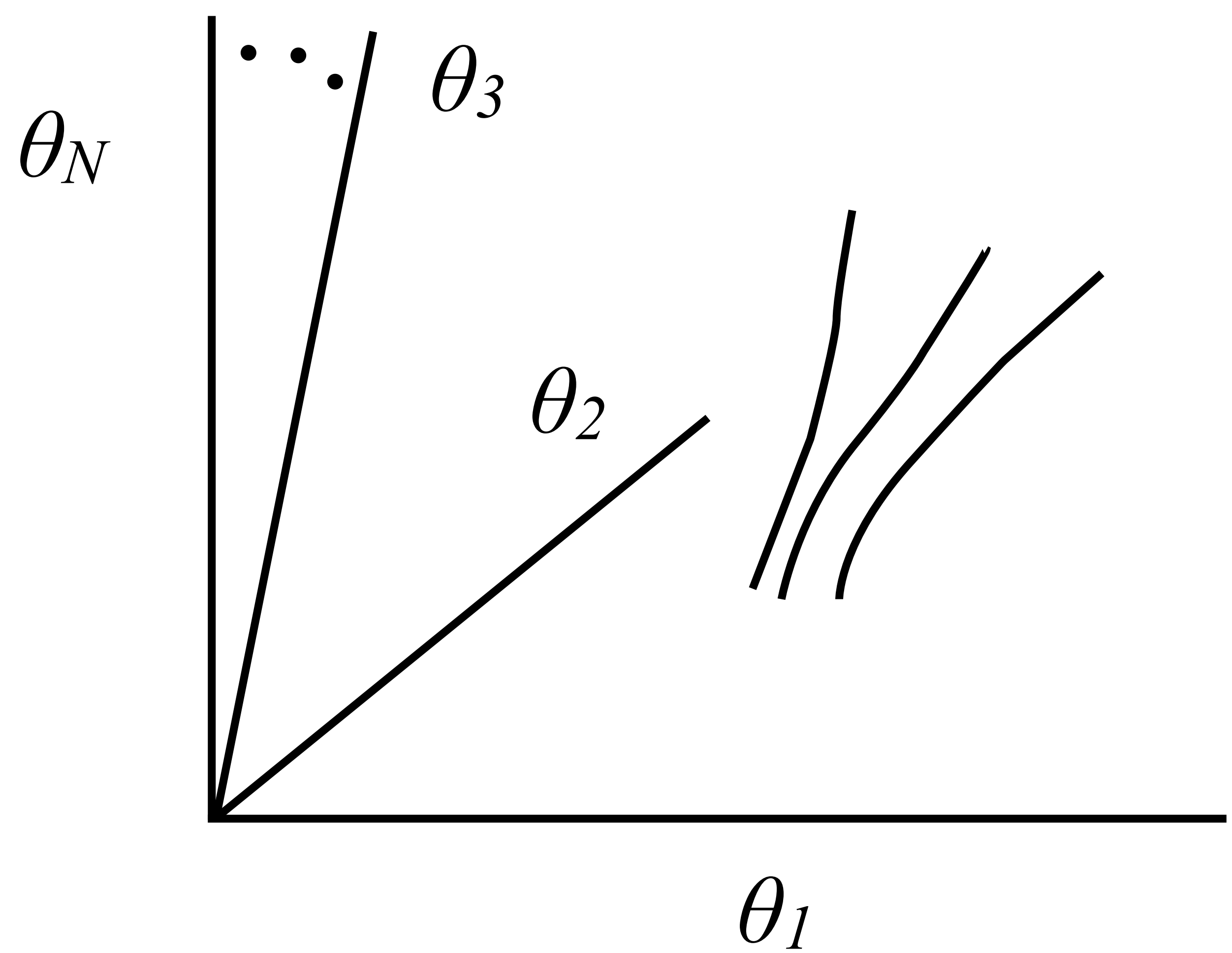
Probability density evolution in N dimensions



Different initial data, parameters

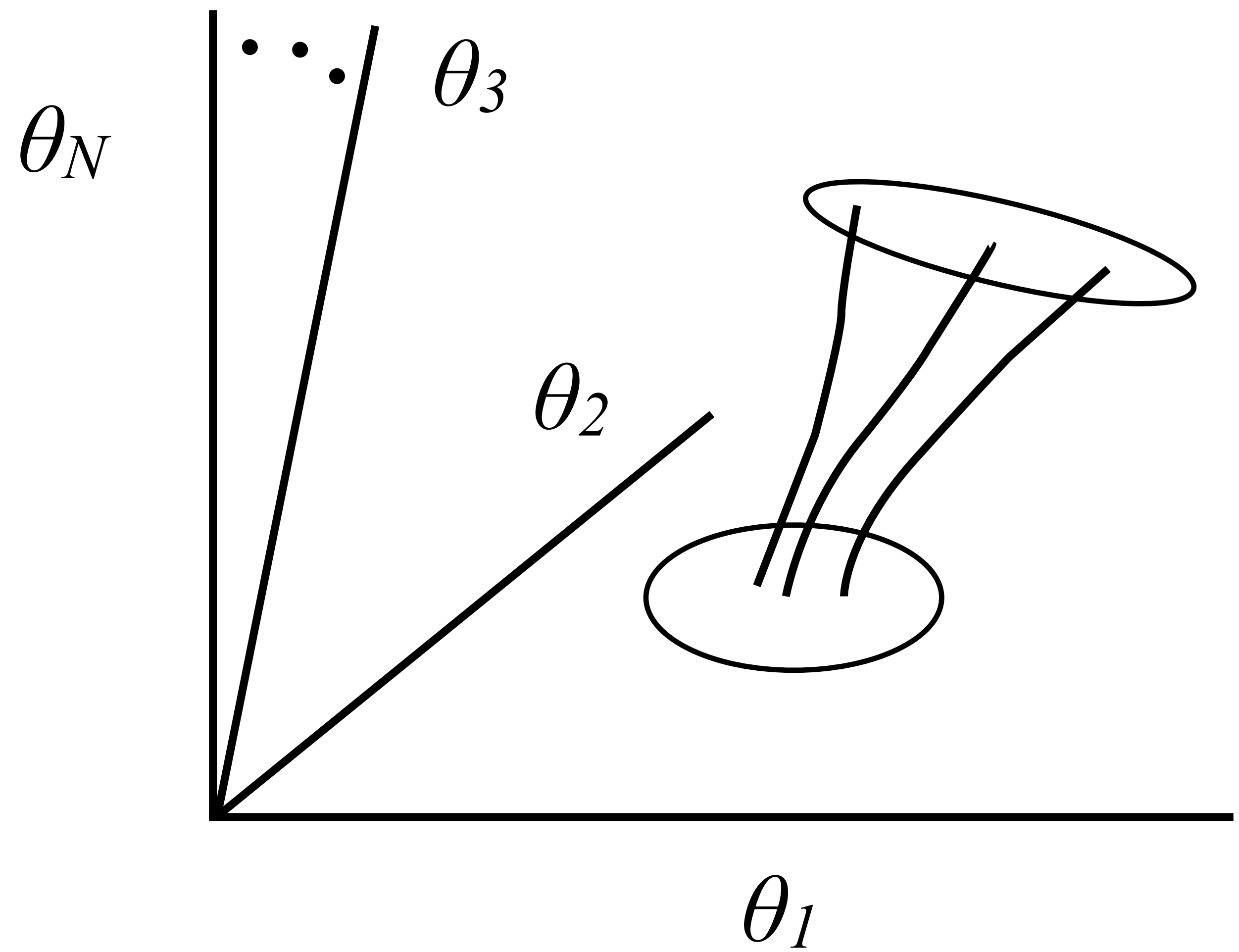
Probability density evolution in N dimensions

Different initial data, parameters



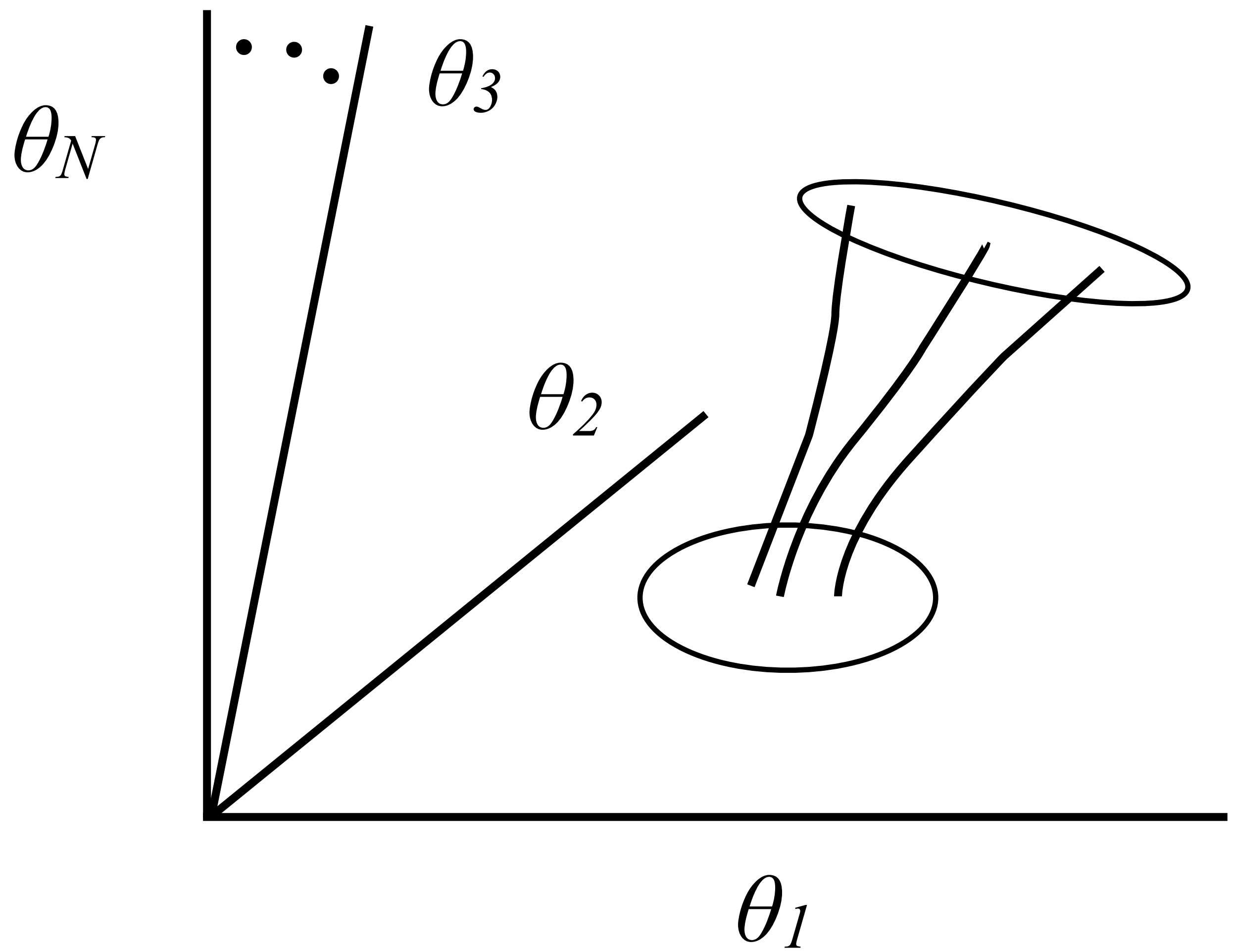
Probability density evolution in N dimensions

Different initial data, parameters



Probability density evolution in N dimensions

Different initial data, parameters



$P_N(\vec{\theta}, t)$

Liouville formalism

$$\dot{\theta}_i = f_i(\vec{\theta}, t) \qquad \vec{\theta} = \{\theta_1, \theta_2, \ldots, \theta_N\}$$

Liouville formalism

$$\dot{\theta}_i = f_i(\vec{\theta}, t) \qquad \vec{\theta} = \{\theta_1, \theta_2, \ldots, \theta_N\}$$

Probability conservation

Liouville formalism

$$\dot{\theta}_i = f_i(\vec{\theta}, t) \qquad \vec{\theta} = \{\theta_1, \theta_2, \ldots, \theta_N\}$$

Probability conservation

$$\frac{\partial P_N(\vec{\theta})}{\partial t} = -\frac{\partial}{\partial \theta_i} f_i P_N(\vec{\theta})$$

Liouville formalism

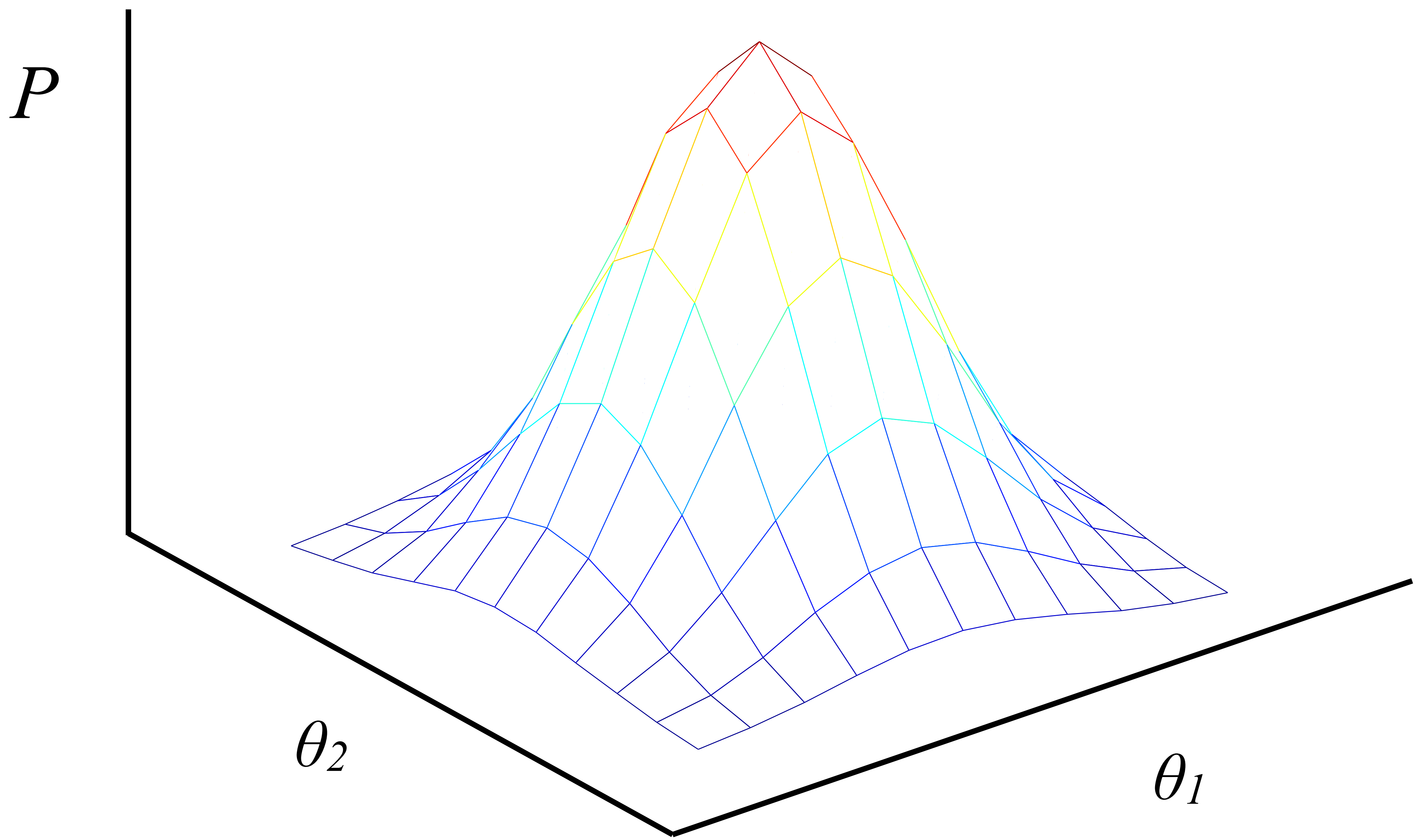
$$\dot{\theta}_i = f_i(\vec{\theta}, t) \qquad \vec{\theta} = \{\theta_1, \theta_2, \ldots, \theta_N\}$$

Probability conservation

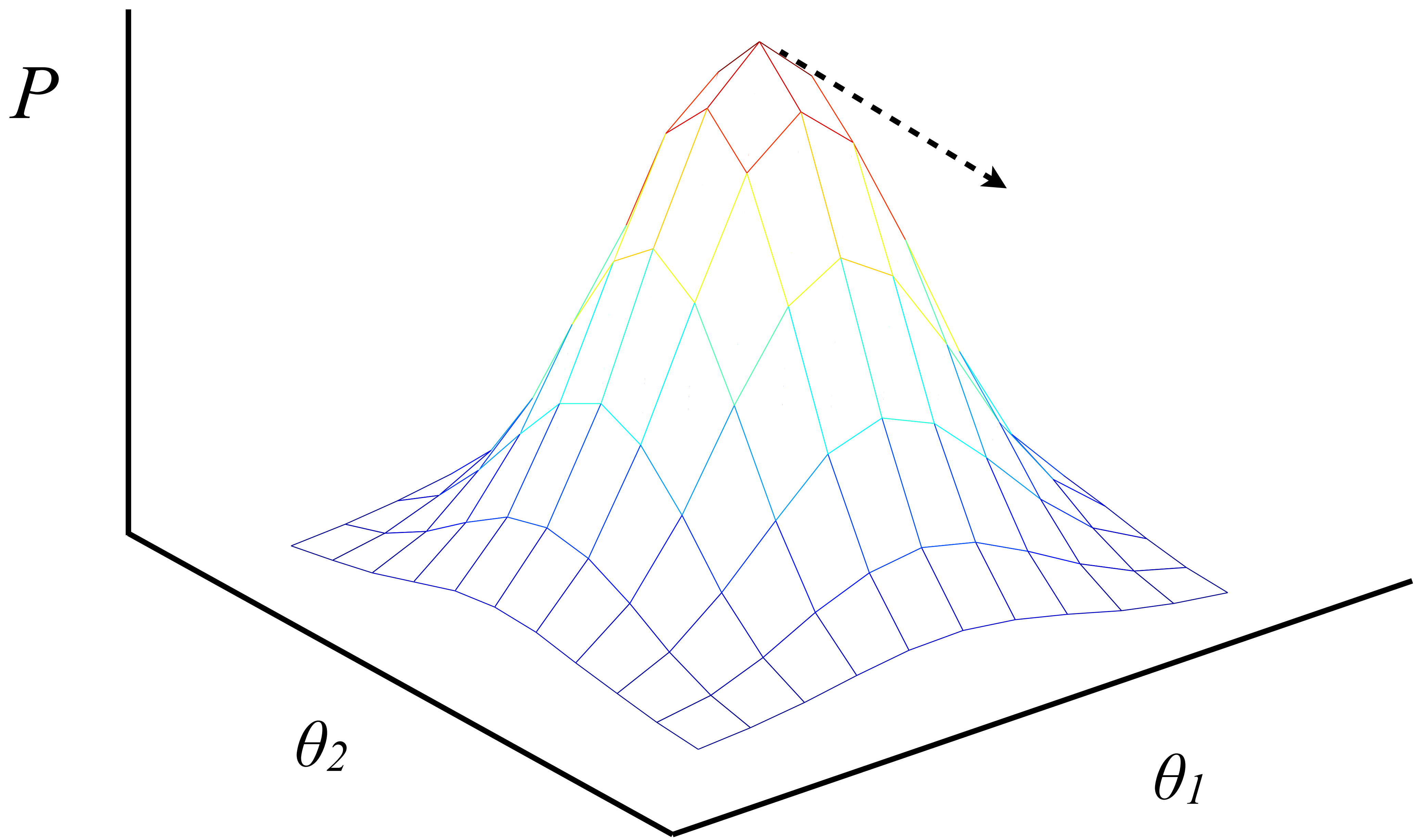
$$\frac{\partial P_N(\vec{\theta})}{\partial t} = -\frac{\partial}{\partial \theta_i} f_i P_N(\vec{\theta})$$

(Einstein summation convention)

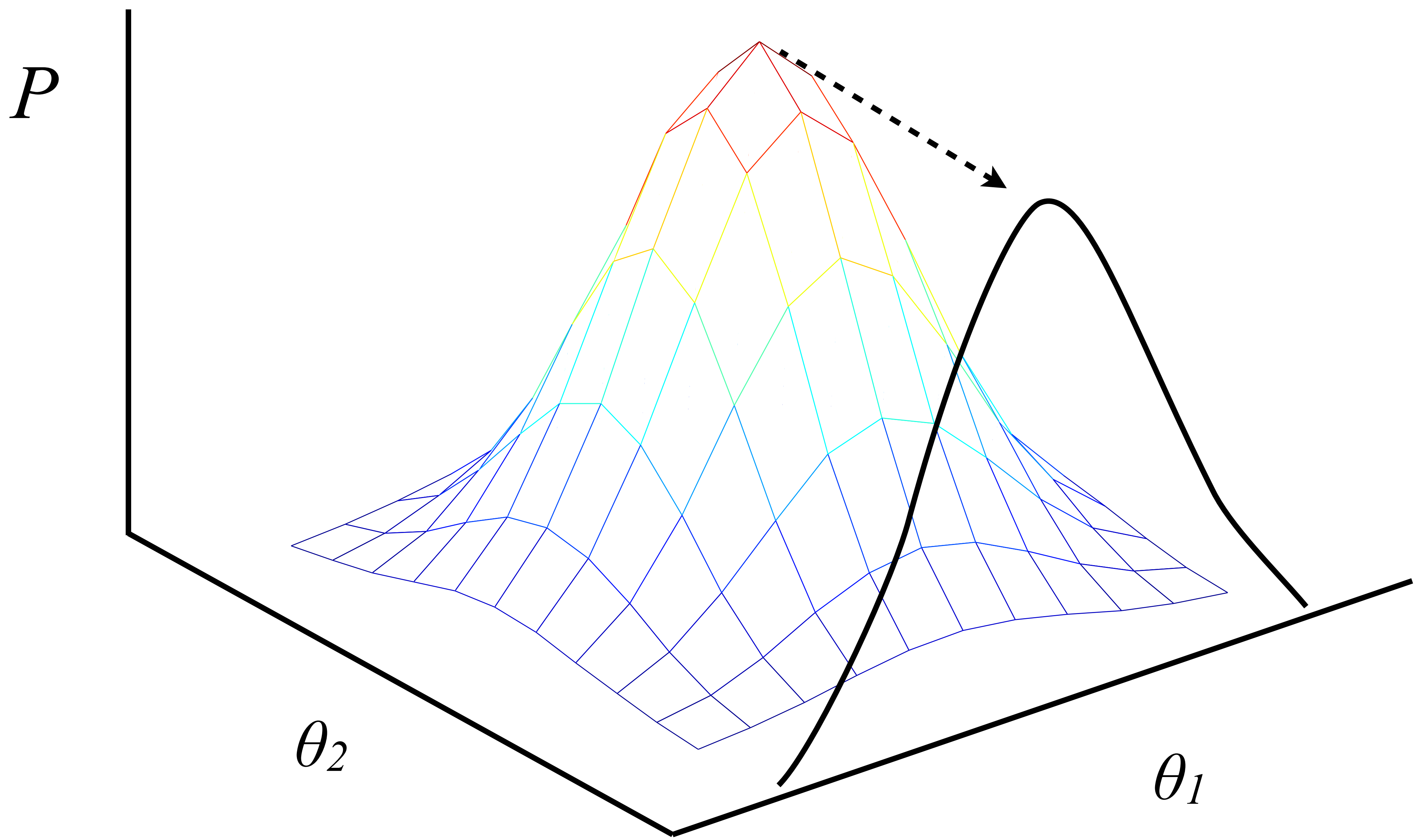
Marginalize



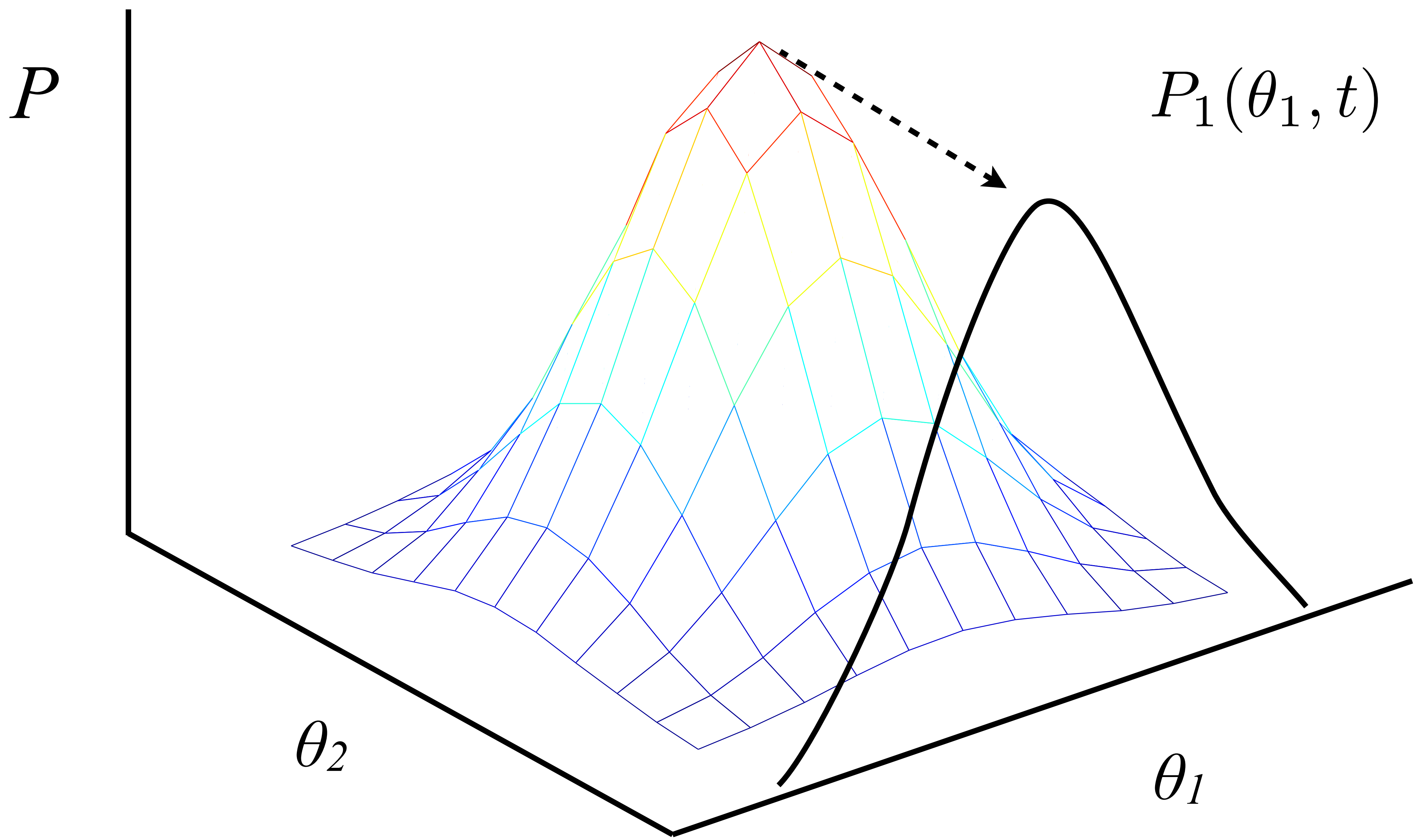
Marginalize



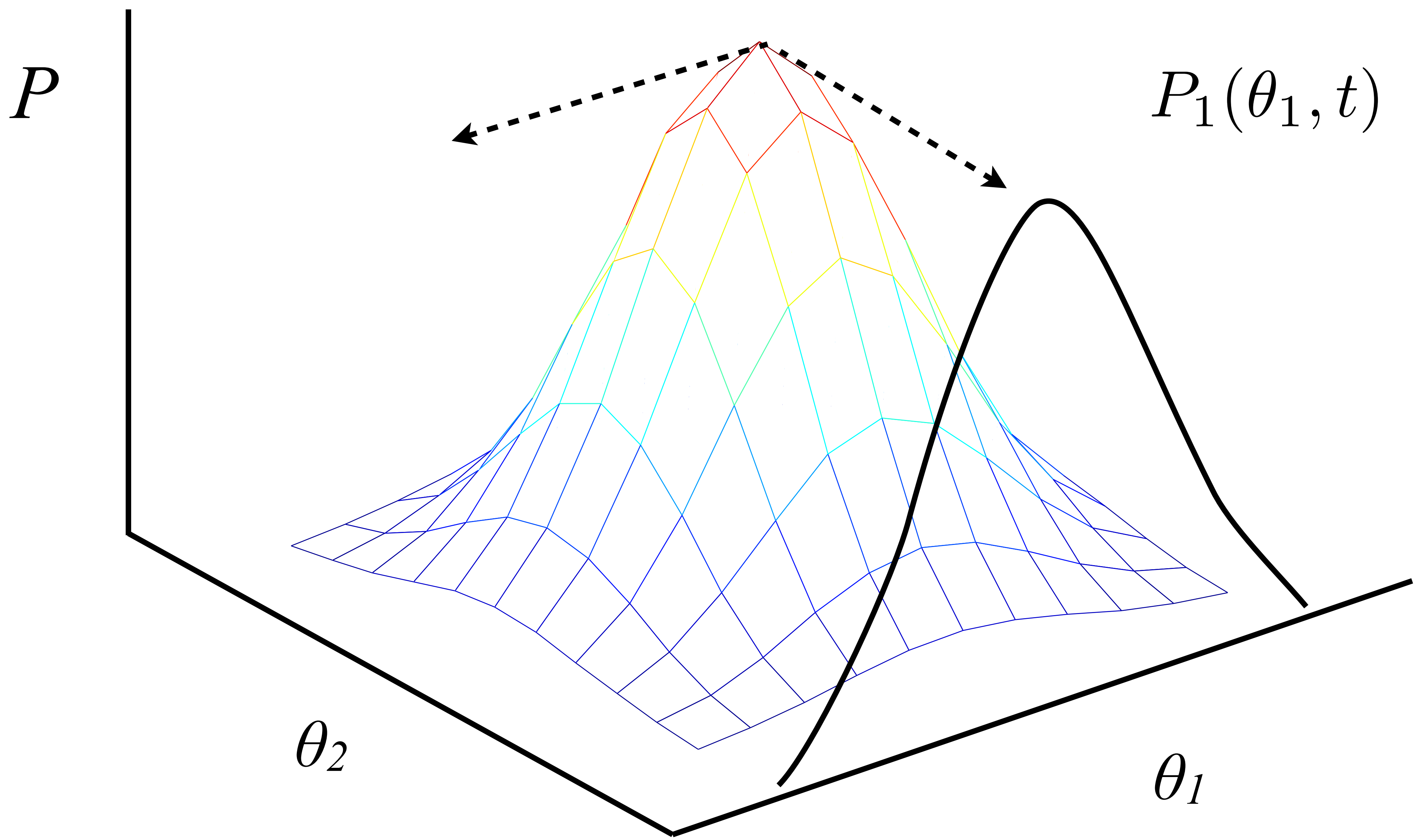
Marginalize



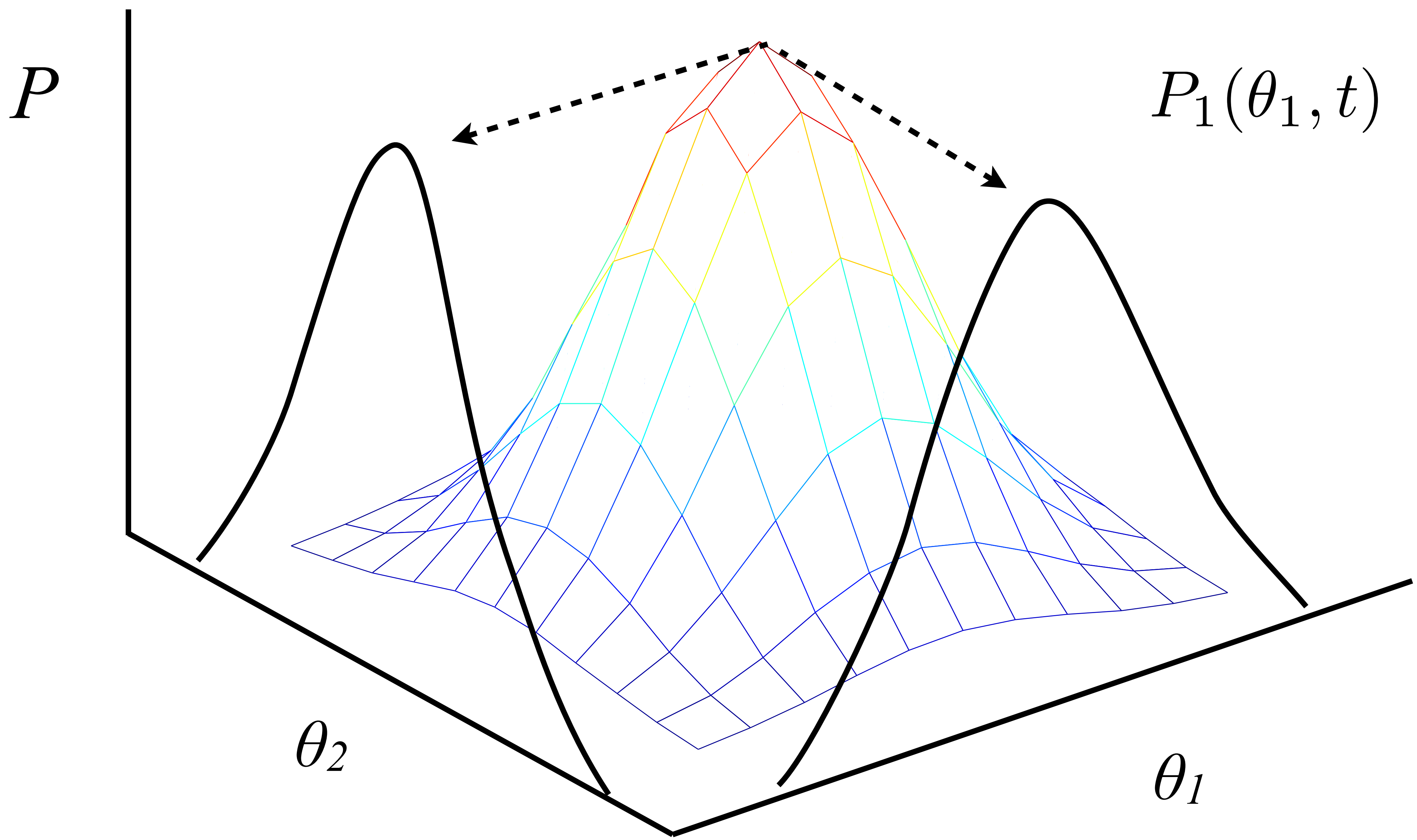
Marginalize



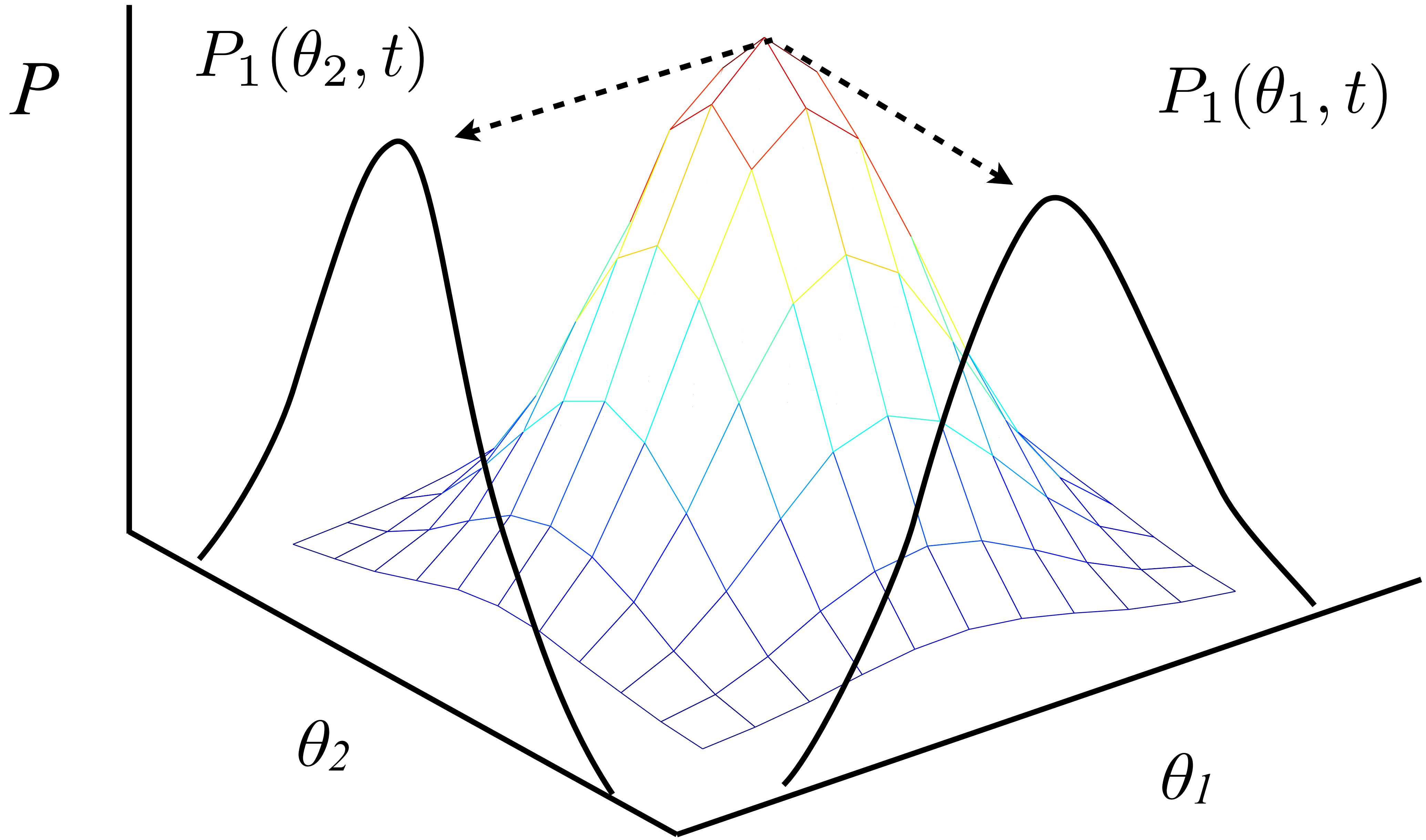
Marginalize



Marginalize



Marginalize



Marginalize

$$P_k(\theta_1, \cdots, \theta_k) = \int \prod_{i=k+1}^{N} d\theta_i P_N(\vec{\theta})$$

Marginalize

$$P_k(\theta_1, \cdots, \theta_k) = \int \prod_{i=k+1}^{N} d\theta_i P_N(\vec{\theta})$$

Exchangeability

Marginalize

$$P_k(\theta_1, \cdots, \theta_k) = \int \prod_{i=k+1}^{N} d\theta_i P_N(\vec{\theta})$$

Exchangeability $\quad P_N(\cdots, \theta_i, \cdots, \theta_j, \cdots)$

Marginalize

$$P_k(\theta_1, \cdots, \theta_k) = \int \prod_{i=k+1}^{N} d\theta_i P_N(\vec{\theta})$$

Exchangeability $$P_N(\cdots, \theta_i, \cdots, \theta_j, \cdots)$$

Marginalize

$$P_k(\theta_1, \cdots, \theta_k) = \int \prod_{i=k+1}^{N} d\theta_i P_N(\vec{\theta})$$

Exchangeability $\quad P_N(\cdots, \theta_i, \cdots, \theta_j, \cdots)$

$$P_1(\theta_1) = P_1(\theta_2) = \cdots = P_1(\theta)$$

$$\frac{\partial P_N(\vec{\theta})}{\partial t} = -\frac{\partial}{\partial \theta_i} f_i(\vec{\theta}) P_N(\vec{\theta})$$

$$\int \prod_{i=2}^{N} d\theta_i \;\; \frac{\partial P_N(\vec{\theta})}{\partial t} = -\frac{\partial}{\partial \theta_i} f_i(\vec{\theta}) P_N(\vec{\theta})$$

$$\int \prod_{i=2}^{N} d\theta_i \;\; \frac{\partial P_N(\vec{\theta})}{\partial t} = -\frac{\partial}{\partial \theta_i} f_i(\vec{\theta}) P_N(\vec{\theta})$$

For pairwise interactions, e.g. $f_i(\vec{\theta}) = \sum_{j=1}^{N} f(\theta_i, \theta_j)$

$$\int \prod_{i=2}^{N} d\theta_i \ \ \frac{\partial P_N(\vec{\theta})}{\partial t} = -\frac{\partial}{\partial \theta_i} f_i(\vec{\theta}) P_N(\vec{\theta})$$

For pairwise interactions, e.g. $f_i(\vec{\theta}) = \sum_{j=1}^{N} f(\theta_i, \theta_j)$

$$\frac{\partial P_1(\theta)}{\partial t} = -N \frac{\partial}{\partial \theta} \int d\theta' f(\theta, \theta') P_2(\theta, \theta')$$

$$P_2(\theta, \theta') = P_1(\theta')P_1(\theta) + \frac{1}{N}C_2(\theta, \theta')$$

$$P_2(\theta,\theta') = P_1(\theta')P_1(\theta) + \frac{1}{N}C_2(\theta,\theta')$$

$$\frac{\partial P_1(\theta)}{\partial t} + N\frac{\partial}{\partial\theta}\int d\theta' f(\theta,\theta')P_1(\theta')P_1(\theta) = -\frac{\partial}{\partial\theta}\int d\theta' f(\theta,\theta')C_2(\theta,\theta')$$

$$P_2(\theta, \theta') = P_1(\theta')P_1(\theta) + \frac{1}{N}C_2(\theta, \theta')$$

$$\frac{\partial P_1(\theta)}{\partial t} + N\frac{\partial}{\partial \theta}\int d\theta' f(\theta, \theta')P_1(\theta')P_1(\theta) = -\frac{\partial}{\partial \theta}\int d\theta' f(\theta, \theta')C_2(\theta, \theta')$$

Finite size effects

$$P_2(\theta,\theta') = P_1(\theta')P_1(\theta) + \frac{1}{N}C_2(\theta,\theta')$$

$$\frac{\partial P_1(\theta)}{\partial t} + N\frac{\partial}{\partial\theta}\int d\theta' f(\theta,\theta')P_1(\theta')P_1(\theta) = -\frac{\partial}{\partial\theta}\int d\theta' f(\theta,\theta')C_2(\theta,\theta')$$

Finite size effects

$$\frac{\partial P_1(\theta)}{\partial t} + N\frac{\partial}{\partial\theta}\int d\theta' f(\theta,\theta')P_1(\theta')P_1(\theta) = 0$$

Mean field theory

$$P_2(\theta, \theta') = P_1(\theta')P_1(\theta) + \frac{1}{N} C_2(\theta, \theta')$$

$$\frac{\partial P_1(\theta)}{\partial t} + N \frac{\partial}{\partial \theta} \int d\theta' f(\theta, \theta') P_1(\theta') P_1(\theta) = -\frac{\partial}{\partial \theta} \int d\theta' f(\theta, \theta') C_2(\theta, \theta')$$

Finite size effects

$$\frac{\partial P_1(\theta)}{\partial t} + N \frac{\partial}{\partial \theta} \int d\theta' f(\theta, \theta') P_1(\theta') P_1(\theta) = 0$$

Mean field theory   Vlasov equation

BBGKY Hierarchy

$$\frac{\partial P_1(\theta)}{\partial t} + N \frac{\partial}{\partial \theta} \int d\theta' f_i(\theta, \theta') P_1(\theta') P_1(\theta) = -N \frac{\partial}{\partial \theta} \int d\theta' f_i(\theta, \theta') C_2(\theta, \theta')$$

BBGKY Hierarchy

$$\frac{\partial P_1(\theta)}{\partial t} + N\frac{\partial}{\partial \theta}\int d\theta' f_i(\theta,\theta')P_1(\theta')P_1(\theta) = -N\frac{\partial}{\partial \theta}\int d\theta' f_i(\theta,\theta')C_2(\theta,\theta')$$

C_2 depends on C_3 and so on

BBGKY Hierarchy

$$\frac{\partial P_1(\theta)}{\partial t} + N\frac{\partial}{\partial\theta}\int d\theta' f_i(\theta,\theta')P_1(\theta')P_1(\theta) = -N\frac{\partial}{\partial\theta}\int d\theta' f_i(\theta,\theta')C_2(\theta,\theta')$$

C_2 depends on C_3 and so on

N coupled PDEs

BBGKY Hierarchy

$$\frac{\partial P_1(\theta)}{\partial t} + N\frac{\partial}{\partial \theta}\int d\theta' f_i(\theta,\theta')P_1(\theta')P_1(\theta) = -N\frac{\partial}{\partial \theta}\int d\theta' f_i(\theta,\theta')C_2(\theta,\theta')$$

C_2 depends on C_3 and so on

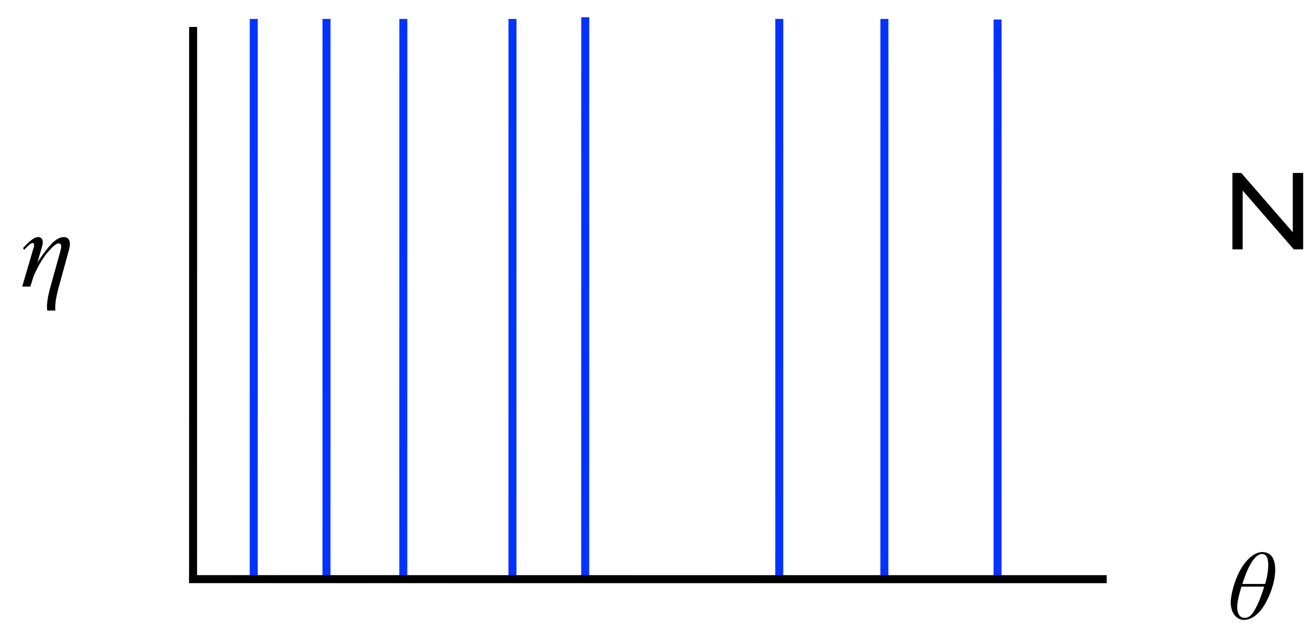
N coupled PDEs

Need to truncate

Exploiting exchangeability

$$P_N(\ldots,\theta_i,\ldots,\theta_j,\ldots) = P_N(\ldots,\theta_j,\ldots,\theta_i,\ldots)$$

Neuron identity is unimportant

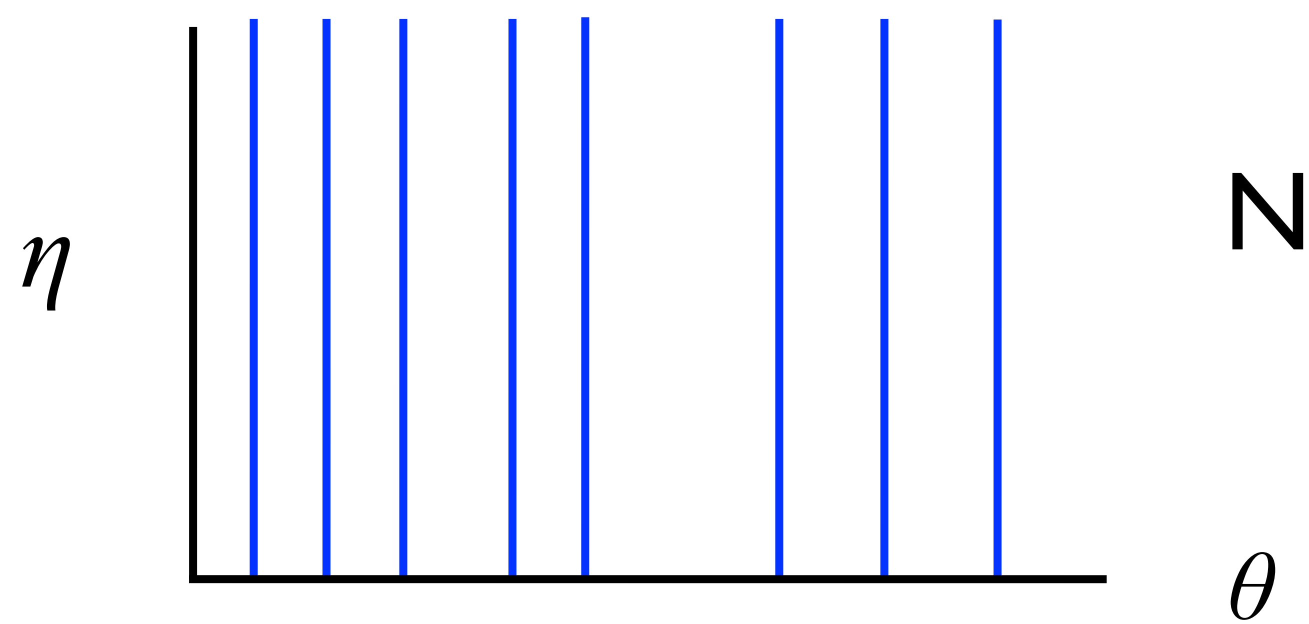


N neurons in 1^{D}

Exploiting exchangeability

$$P_N(\ldots,\theta_i,\ldots,\theta_j,\ldots) = P_N(\ldots,\theta_j,\ldots,\theta_i,\ldots)$$

Neuron identity is unimportant



N neurons in 1^{D}

density $$\eta(\theta,u,t) = \frac{1}{N}\sum_{i=1}^{N}\delta(\theta-\theta_i(t))$$

Apply to phase neuron model

Neuron dynamics: $\dot{\theta} = I(t) + \alpha u(t)$

Apply to phase neuron model

Neuron dynamics: $\dot{\theta} = I(t) + \alpha u(t)$

Synaptic dynamics:

Apply to phase neuron model

Neuron dynamics: $$\dot{\theta} = I(t) + \alpha u(t)$$

Synaptic dynamics: $$\dot{u} + \beta u = \beta \nu$$

Apply to phase neuron model

Neuron dynamics: $$\dot{\theta} = I(t) + \alpha u(t)$$

Synaptic dynamics: $$\dot{u} + \beta u = \beta \nu$$

Firing rate:

Apply to phase neuron model

Neuron dynamics: $\dot{\theta} = I(t) + \alpha u(t)$

Synaptic dynamics: $\dot{u} + \beta u = \beta \nu$

Firing rate: $\nu = \frac{\beta}{N} \sum_j \delta(t - t_j^s)$

Firing rate: $$\nu = \frac{\beta}{N} \sum_j \delta(t - t_j^s)$$

density: $$\eta(\theta, u, t) = \frac{1}{N} \sum_{i=1}^{N} \delta(\theta - \theta_i(t))$$

Firing rate: $$\nu = \frac{\beta}{N}\sum_j \delta(t - t_j^s)$$

density: $$\eta(\theta, u, t) = \frac{1}{N}\sum_{i=1}^{N} \delta(\theta - \theta_i(t))$$

$$\delta(t - t_j^s) = \dot{\theta}\delta(\pi - \theta(t))$$

Firing rate: $$\nu = \frac{\beta}{N} \sum_j \delta(t - t_j^s)$$

density: $$\eta(\theta, u, t) = \frac{1}{N} \sum_{i=1}^{N} \delta(\theta - \theta_i(t))$$

$$\delta(t - t_j^s) = \dot{\theta}\delta(\pi - \theta(t))$$

$$\nu(t) = \frac{1}{N} \sum_i \dot{\theta}_i(t)\delta(\pi - \theta_i(t)) = (I(t) + \alpha u(t))\eta(\pi, t)$$

Klimontovich formalism

e.g. Hildebrand, Buice, Chow, PRL 98.054101, 2007

Complete description of system

$$\partial_t \eta + \partial_\theta \left[(I(t) + \alpha u(t))\eta \right] = 0$$

$$\dot{u} + \beta u = \beta \nu$$

$$\nu(t) = (I(t) + \alpha u(t))\eta(\pi, t)$$

Klimontovich formalism

e.g. Hildebrand, Buice, Chow, PRL 98.054101, 2007

Complete description of system

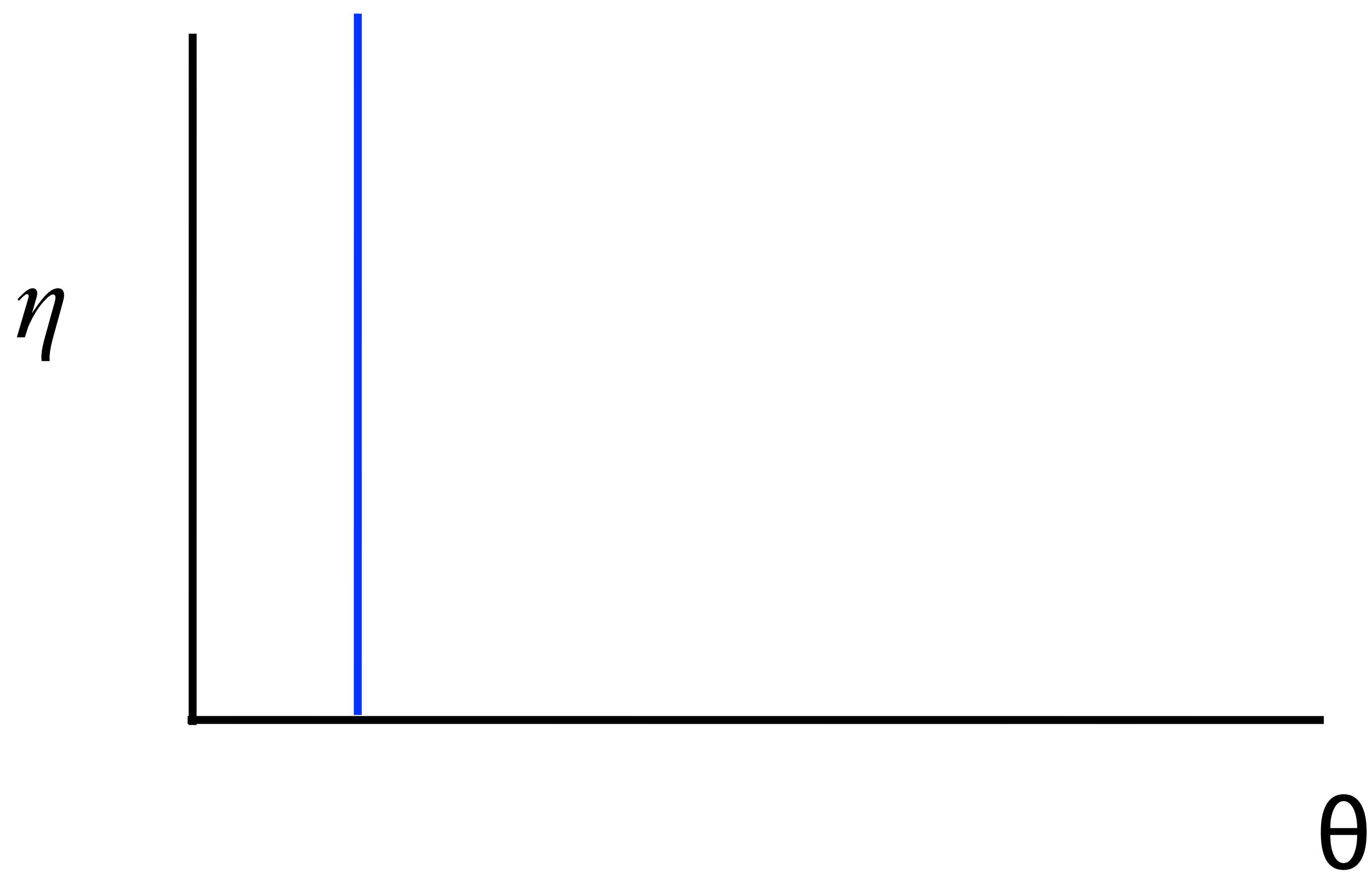
$$\partial_t \eta + \partial_\theta \left[(I(t) + \alpha u(t)) \eta \right] = 0$$

$$\dot{u} + \beta u = \beta \nu$$

$$\nu(t) = (I(t) + \alpha u(t)) \eta(\pi, t)$$

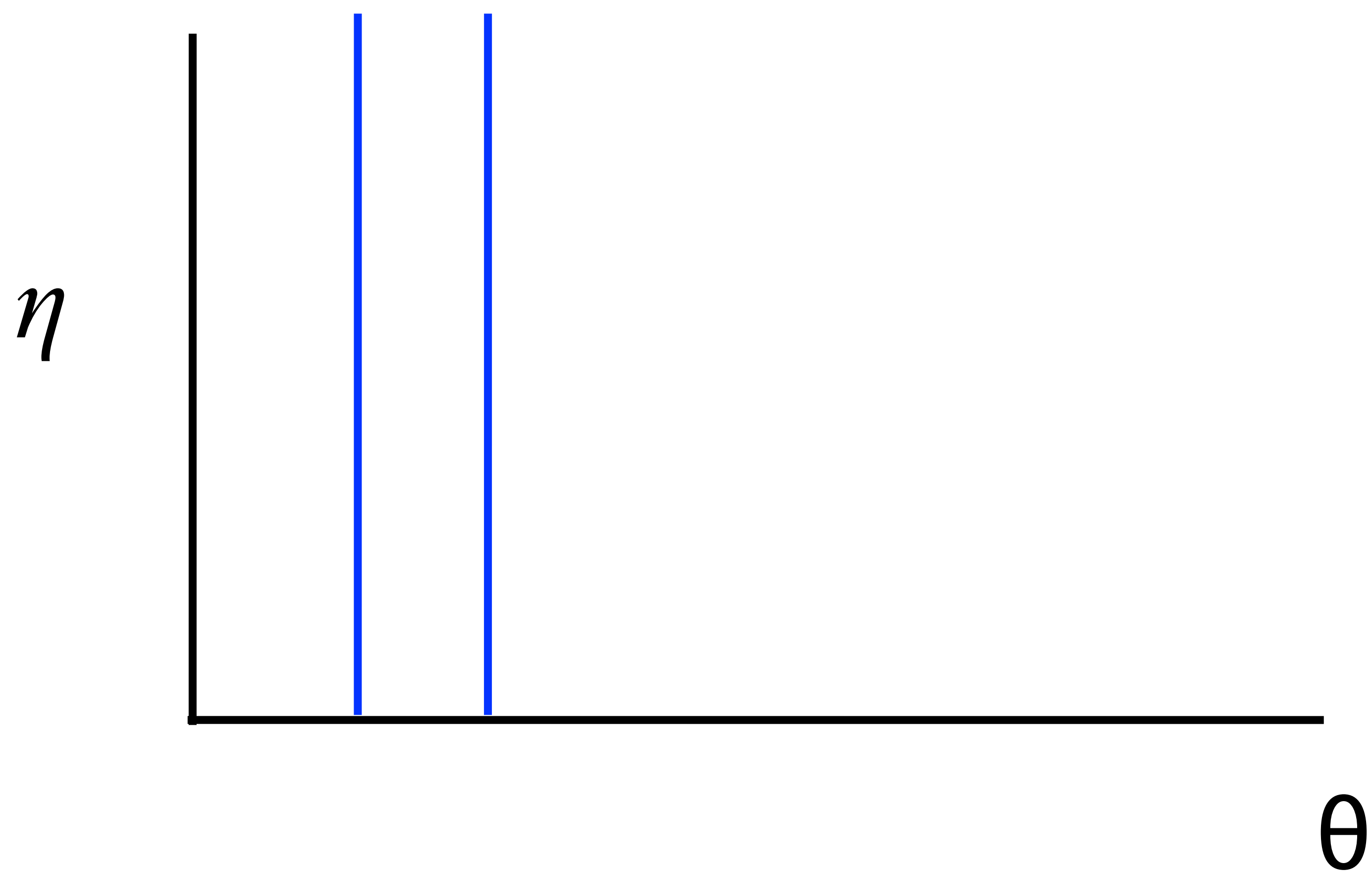
but η is not differentiable

Average over initial data



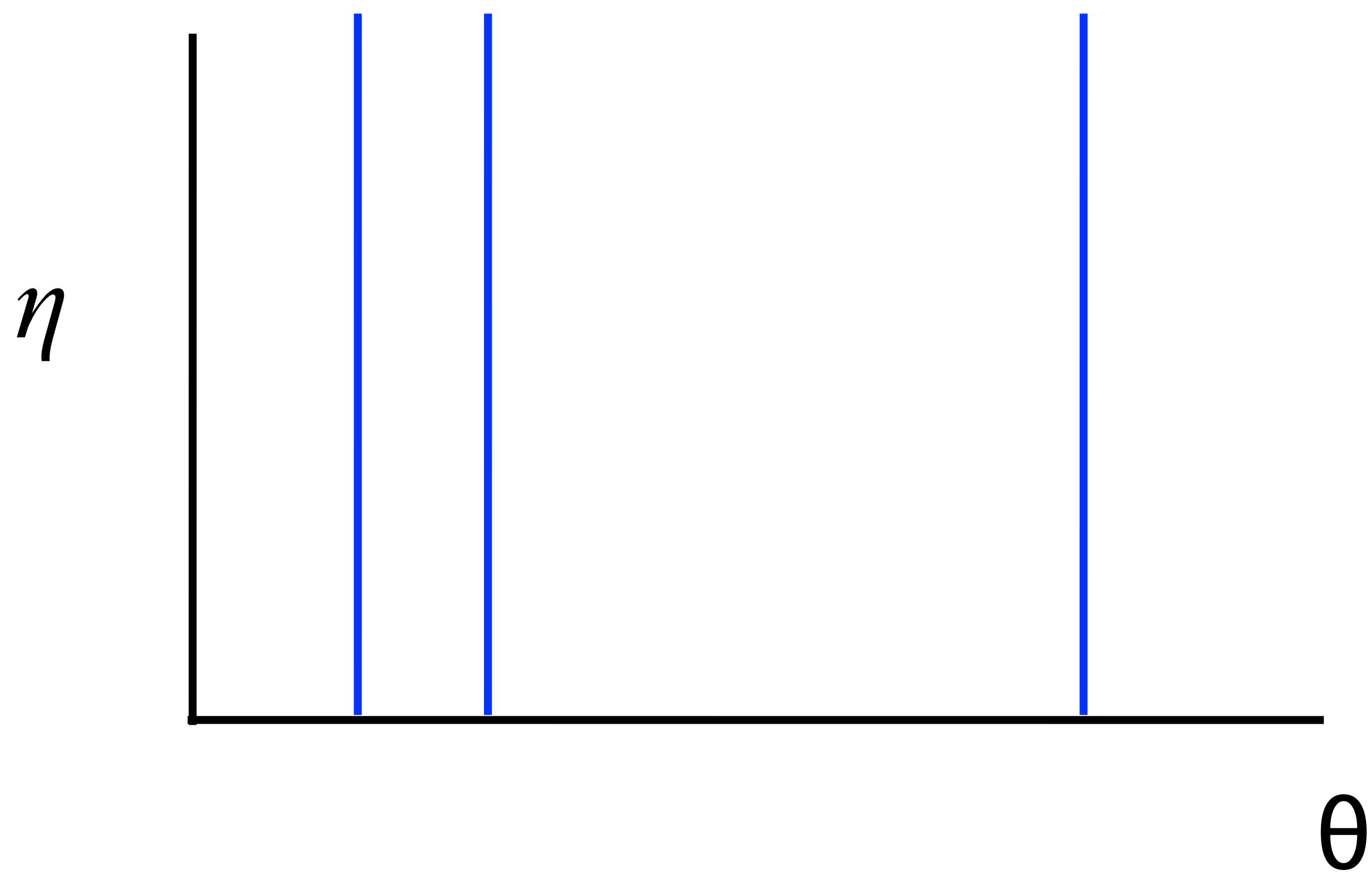
smooth by averaging

Average over initial data



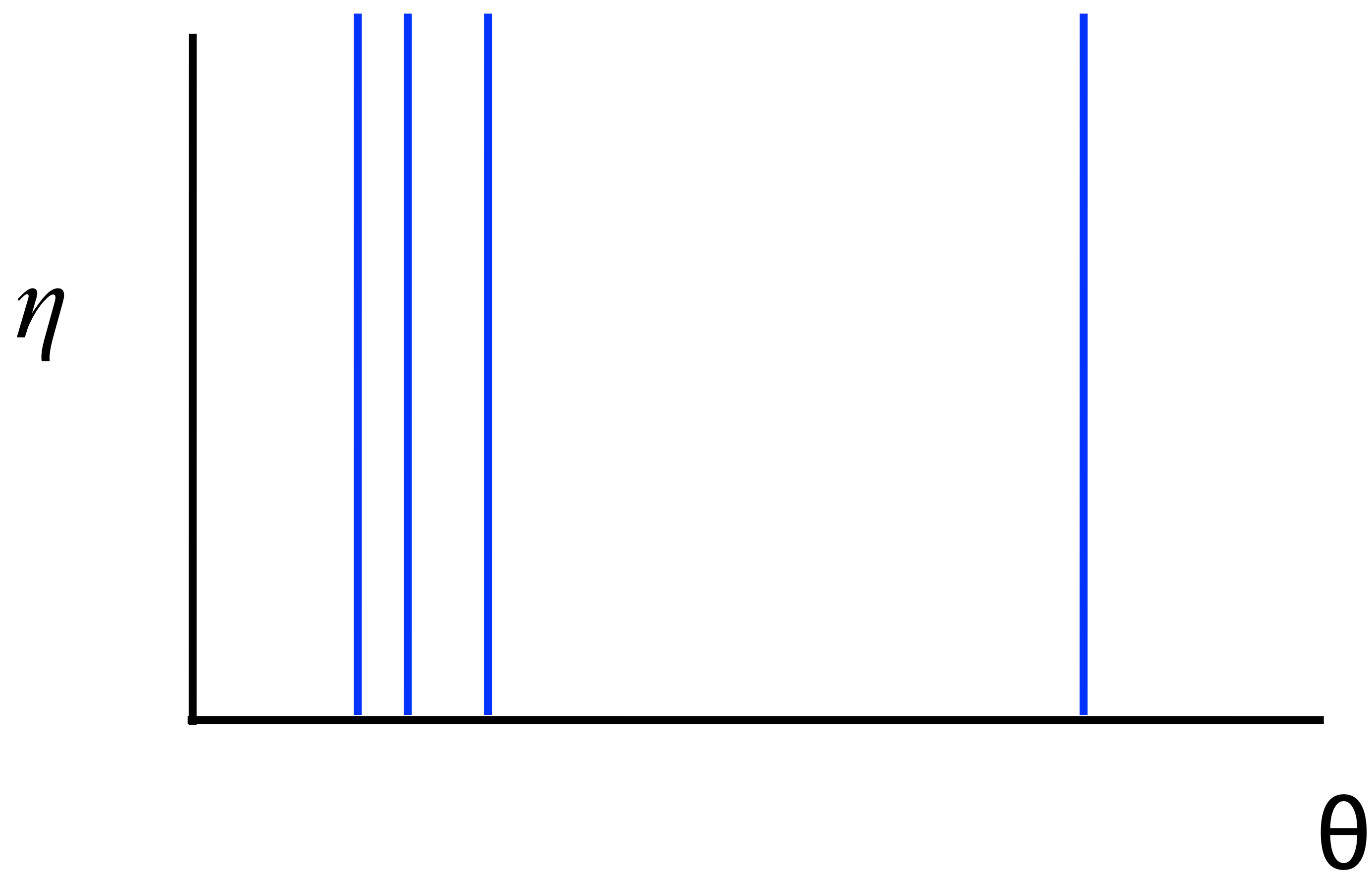
smooth by averaging

Average over initial data



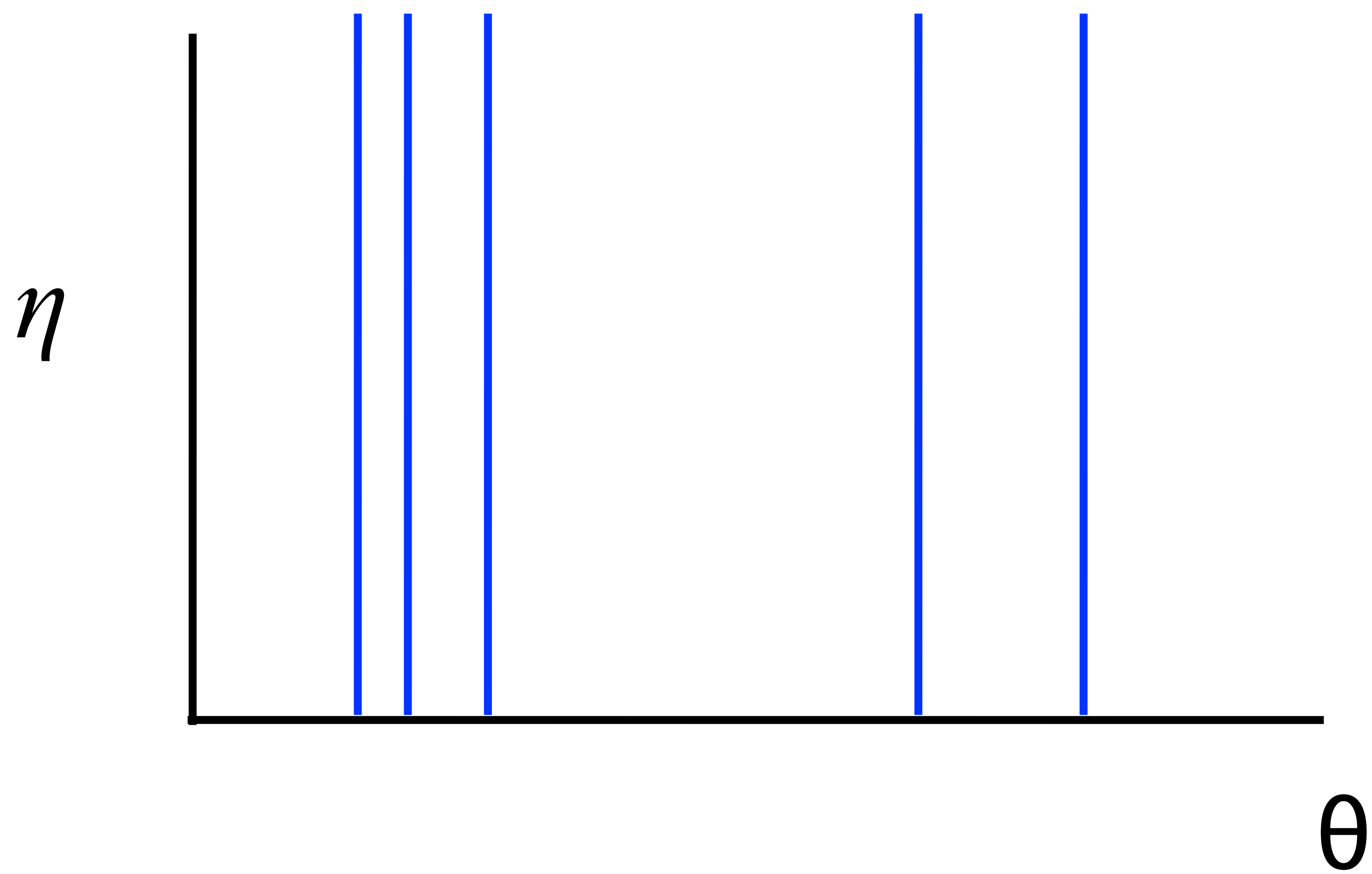
smooth by averaging

Average over initial data



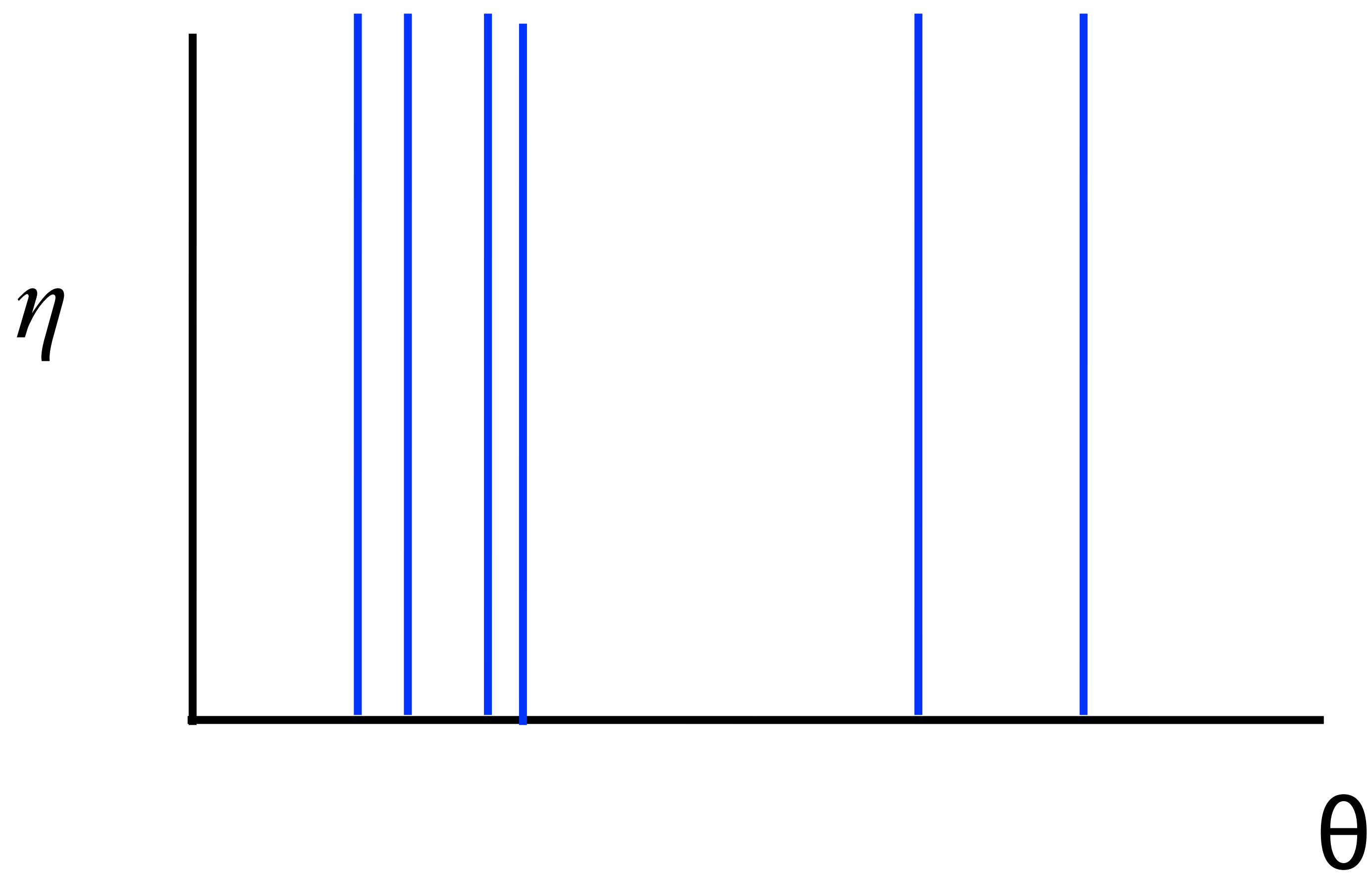
smooth by averaging

Average over initial data



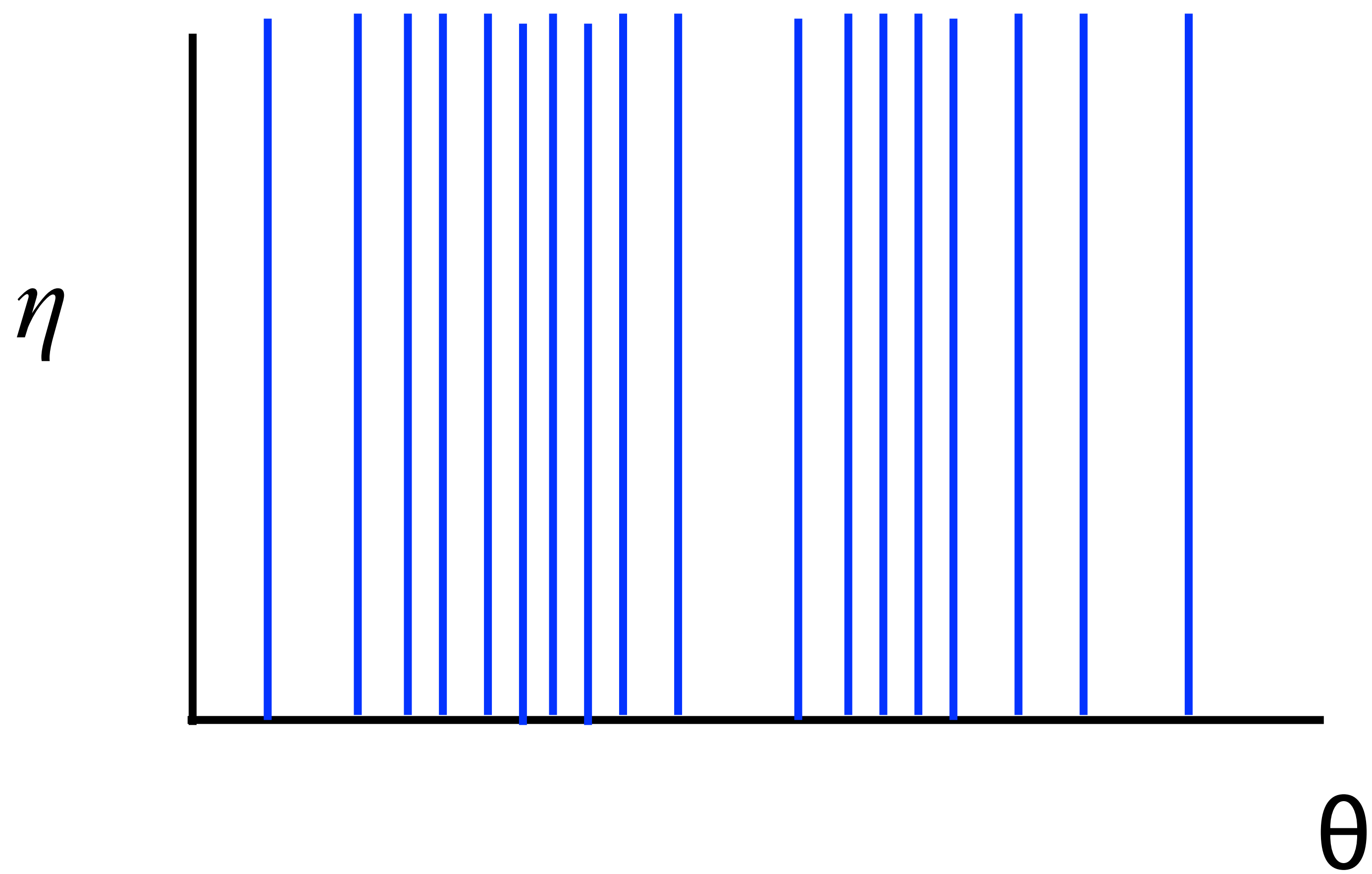
smooth by averaging

Average over initial data



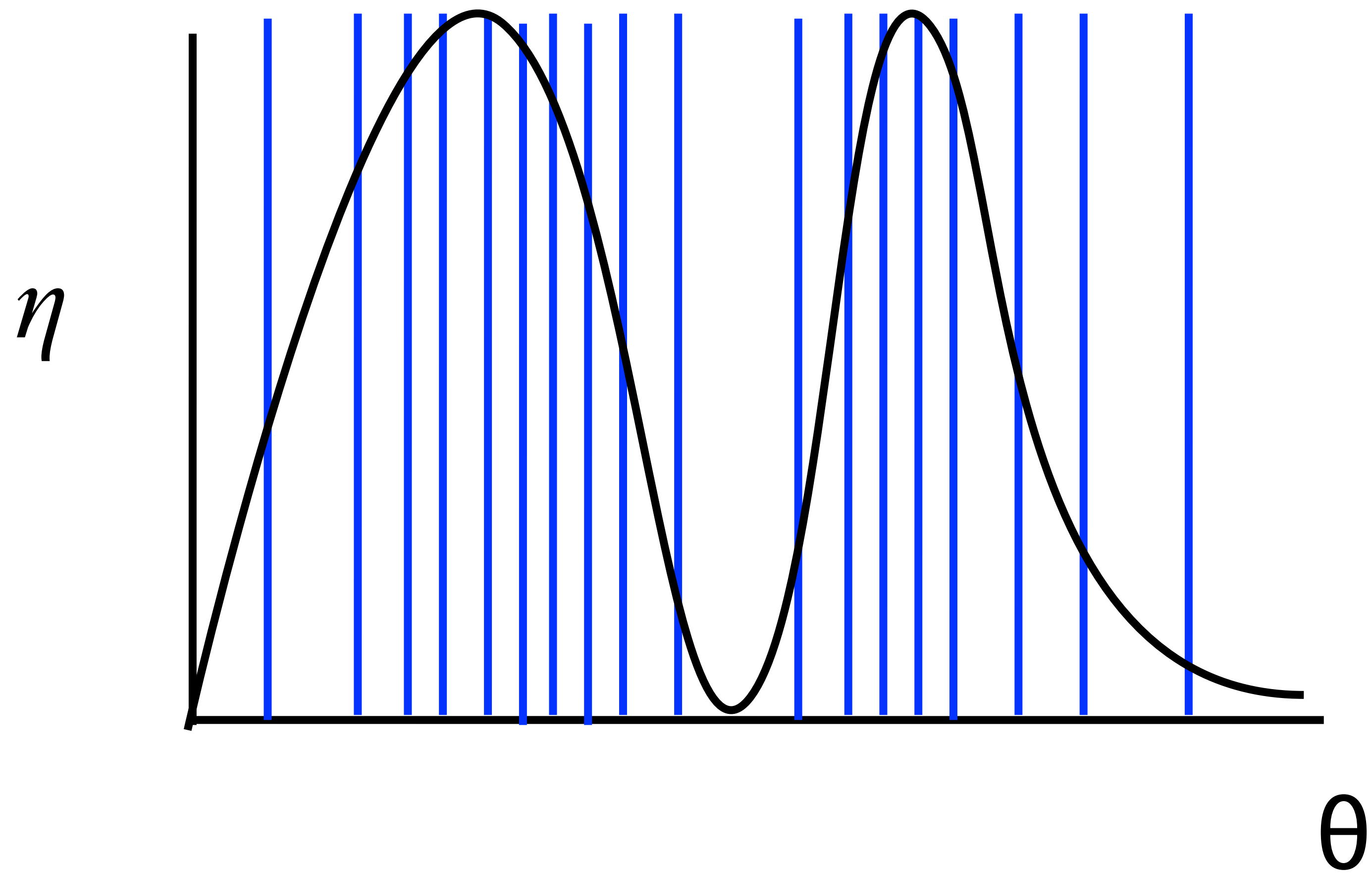
smooth by averaging

Average over initial data



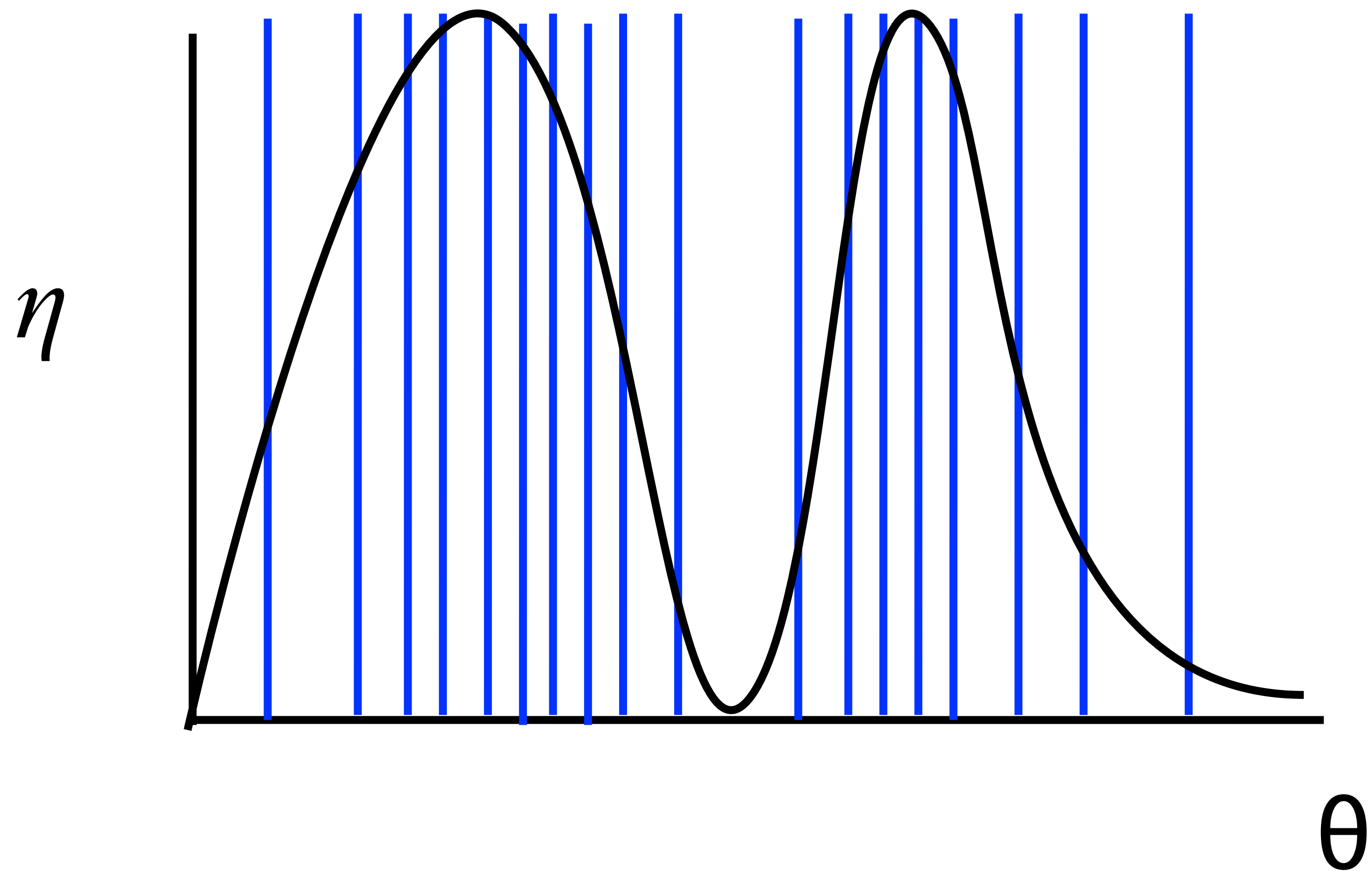
smooth by averaging

Average over initial data



smooth by averaging

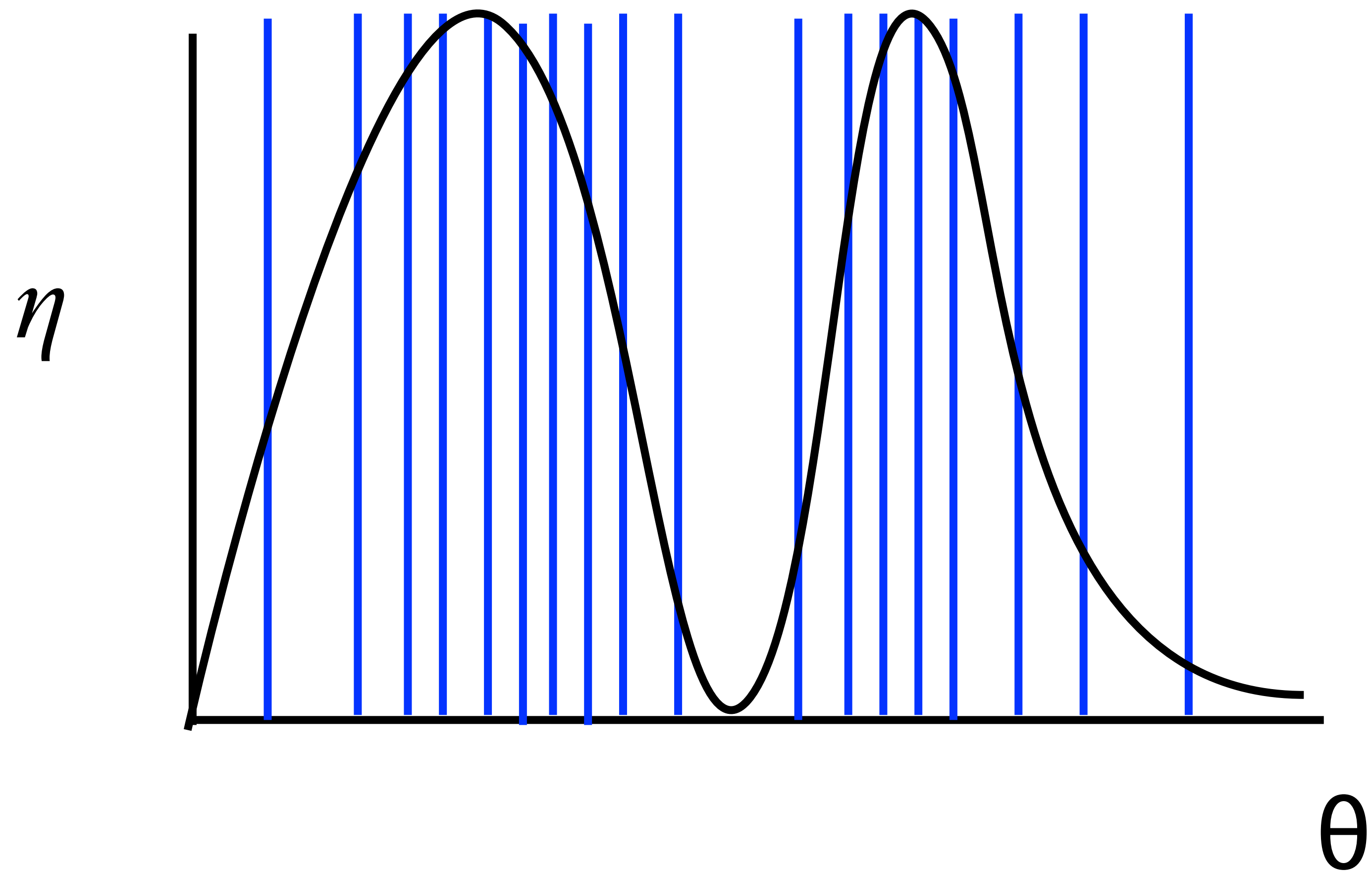
Average over initial data



smooth by averaging

$$\rho(\theta, t) = \langle \eta(\theta, t) \rangle$$

Average over initial data



smooth by averaging

$$\rho(\theta, t) = \langle \eta(\theta, t) \rangle$$

$$u_0 = \langle u \rangle$$

Average over initial data

$$\dot{u}(t) = -\beta u(t) + \beta \left[I(t)\eta + \alpha u \eta \right]$$

Average over initial data

$$\left\langle \; \dot{u}(t) = -\beta u(t) + \beta \left[I(t)\eta + \alpha u \eta \right] \; \right\rangle$$

Average over initial data

$$\dot{u}_0(t) = -\beta u_0(t) + \beta \left[I(t)\rho + \alpha \langle u\eta \rangle \right]$$

Average over initial data

$$\dot{u}_0(t) = -\beta u_0(t) + \beta \left[I(t)\rho + \alpha \langle u\eta \rangle \right]$$

$$\partial_t \eta + \partial_\theta \left[I(t)\eta + \alpha u \eta \right] = 0$$

Average over initial data

$$\dot{u}_0(t) = -\beta u_0(t) + \beta \left[I(t)\rho + \alpha \langle u\eta \rangle \right]$$

$$\Big\langle \; \partial_t \eta + \partial_\theta \left[I(t)\eta + \alpha u \eta \right] = 0 \; \Big\rangle$$

Average over initial data

$$\dot{u}_0(t) = -\beta u_0(t) + \beta\left[I(t)\rho + \alpha\langle u\eta\rangle\right]$$

$$\partial_t \rho + \partial_\theta\left[I(t)\rho + \alpha\langle u\eta\rangle\right] = 0$$

Average over initial data

$$\dot{u}_0(t) = -\beta u_0(t) + \beta \left[I(t)\rho + \alpha \langle u\eta \rangle \right]$$

$$\partial_t \rho + \partial_\theta \left[I(t)\rho + \alpha \langle u\eta \rangle \right] = 0$$

$$\left(\partial_t u(t) + \beta u(t) - \beta \left[I(t)\eta + \alpha u\eta \right] \right) = 0$$

Average over initial data

$$\dot{u}_0(t) = -\beta u_0(t) + \beta\left[I(t)\rho + \alpha\langle u\eta\rangle\right]$$

$$\partial_t \rho + \partial_\theta \left[I(t)\rho + \alpha\langle u\eta\rangle\right] = 0$$

$$\eta\left(\partial_t u(t) + \beta u(t) - \beta\left[I(t)\eta + \alpha u\eta\right]\right) = 0$$

Average over initial data

$$\dot{u}_0(t) = -\beta u_0(t) + \beta \left[I(t)\rho + \alpha \langle u\eta \rangle \right]$$

$$\partial_t \rho + \partial_\theta \left[I(t)\rho + \alpha \langle u\eta \rangle \right] = 0$$

$$\Big\langle \; \eta \left(\partial_t u(t) + \beta u(t) - \beta \left[I(t)\eta + \alpha u\eta \right] \right) = 0 \; \Big\rangle$$

Average over initial data

$$\dot{u}_0(t) = -\beta u_0(t) + \beta \left[I(t)\rho + \alpha \langle u\eta \rangle \right]$$

$$\partial_t \rho + \partial_\theta \left[I(t)\rho + \alpha \langle u\eta \rangle \right] = 0$$

$$\Big\langle \; \eta \left(\partial_t u(t) + \beta u(t) - \beta \left[I(t)\eta + \alpha u\eta \right] \right) = 0 \; \Big\rangle$$

$$\langle \eta u \eta \rangle$$

Average over initial data

$$\dot{u}_0(t) = -\beta u_0(t) + \beta \left[I(t)\rho + \alpha \langle u\eta \rangle \right]$$

$$\partial_t \rho + \partial_\theta \left[I(t)\rho + \alpha \langle u\eta \rangle \right] = 0$$

$$\Big\langle \; \eta \left(\partial_t u(t) + \beta u(t) - \beta \left[I(t)\eta + \alpha u\eta \right] \right) = 0 \; \Big\rangle$$

$$\langle \eta u \eta \rangle$$

$$\vdots$$

Average over initial data

$$\dot{u}_0(t) = -\beta u_0(t) + \beta \left[I(t)\rho + \alpha\langle u\eta\rangle \right]$$

$$\partial_t \rho + \partial_\theta \left[I(t)\rho + \alpha\langle u\eta\rangle \right] = 0$$

$$\Big\langle \; \eta \left(\partial_t u(t) + \beta u(t) - \beta \left[I(t)\eta + \underset{\langle \eta u \eta \rangle}{\alpha u\eta} \right] \right) = 0 \; \Big\rangle$$

$$\vdots$$

BBGKY moment hierarchy

Average over initial data

$$\dot{u}_0(t) = -\beta u_0(t) + \beta \left[I(t)\rho + \alpha \langle u\eta \rangle \right]$$

$$\partial_t \rho + \partial_\theta \left[I(t)\rho + \alpha \langle u\eta \rangle \right] = 0$$

$$\Big\langle \; \eta \left(\partial_t u(t) + \beta u(t) - \beta \left[I(t)\eta + \alpha u\eta \right] \right) = 0 \; \Big\rangle$$

$$\langle \eta u \eta \rangle$$

$$\vdots$$

BBGKY moment hierarchy

$$\langle u\eta \rangle = u_0 \rho + \frac{1}{N} C_{uv}$$

$$\langle u\eta \rangle = u_0 \rho + \frac{1}{N} C_{uv}$$

$$\langle u\eta \rangle = u_0 \rho + \frac{1}{N} C_{uv}$$

Ignore correlations

Mean field theory

$$\langle u\eta \rangle = u_0 \rho + \frac{1}{N} C_{uv}$$

Ignore correlations

Mean field theory

$$\dot{u}_0(t) = -\beta u_0(t) + \beta \nu(t)$$

$$\nu(t) = (I(t) + \alpha u_0(t))\rho(\pi, t)$$

$$\partial_t \rho + \partial_\theta \left[(I(t) + \alpha u_0(t))\rho \right] = 0$$

Mean field theory

$$\dot{u}_0(t) = -\beta u_0(t) + \beta \nu(t)$$

$$\nu(t) = (I(t) + \alpha u_0(t))\rho(\pi, t)$$

$$\partial_t \rho + \partial_\theta \left[(I(t) + \alpha u_0(t))\rho \right] = 0$$

Previous work went straight to mean field theory

e.g. Desai and Zwanzig, 1978; Strogatz and Mirollo, 1990; Treves 1993; Abbott and Van Vreeswijk, 1993; ...

Steady state

$$\dot{u} = -\beta u + \beta(I + \alpha u)\rho(\pi, t) \quad = 0$$

$$\partial_t \rho = -\partial_\theta \left[(I(t) + \alpha u(t))\rho \right] \qquad = 0$$

Steady state

$$\dot{u} = -\beta u + \beta(I + \alpha u)\rho(\pi, t) \quad = 0$$

$$\partial_t \rho = -\partial_\theta \left[(I(t) + \alpha u(t))\rho \right] \qquad = 0$$

$$\bar{\rho} = \frac{1}{2\pi} \qquad\qquad \bar{u} = \frac{I}{2\pi}\left(1 - \frac{\alpha}{2\pi}\right)^{-1}$$

Steady state

$$\dot{u} = -\beta u + \beta(I + \alpha u)\rho(\pi, t) \quad = 0$$

$$\partial_t \rho = -\partial_\theta \left[(I(t) + \alpha u(t))\rho \right] \quad = 0$$

$$\bar{\rho} = \frac{1}{2\pi} \qquad \bar{u} = \frac{I}{2\pi}\left(1 - \frac{\alpha}{2\pi}\right)^{-1}$$

$$\nu = (I + \alpha \bar{u})\bar{\rho} = \bar{u}$$

Activity equation

$$\dot{u} + \beta u = \beta(I + \alpha u)\rho(\pi, t)$$

$$\partial_t \rho + \partial_\theta (I + \alpha u(t))\rho(\theta, t)) = \rho_0(\theta)\delta(t)$$

Activity equation

$$\dot{u} + \beta u = \beta(I + \alpha u)\rho(\pi, t)$$

$$\partial_t \rho + \partial_\theta (I + \alpha u(t))\rho(\theta, t)) = \rho_0(\theta)\delta(t)$$

$$\dot{u} + \beta u = \beta(I + \alpha u)\rho_0 \left(\pi - It - \alpha \int_0^t u(s)ds \right)$$

Activity equation

$$\dot{u} + \beta u = \beta(I + \alpha u)\rho(\pi, t)$$

$$\partial_t \rho + \partial_\theta (I + \alpha u(t))\rho(\theta, t)) = \rho_0(\theta)\delta(t)$$

$$\dot{u} + \beta u = \beta(I + \alpha u)\rho_0 \left(\pi - It - \alpha \int_0^t u(s)ds \right)$$

If $\rho_0 = \dfrac{1}{2\pi}$

Activity equation

$$\dot{u} + \beta u = \beta(I + \alpha u)\rho(\pi, t)$$

$$\partial_t \rho + \partial_\theta (I + \alpha u(t))\rho(\theta, t)) = \rho_0(\theta)\delta(t)$$

$$\dot{u} + \beta u =$$

If $\rho_0 = \frac{1}{2\pi}$

Activity equation

$$\dot{u} + \beta u = \beta(I + \alpha u)\rho(\pi, t)$$

$$\partial_t \rho + \partial_\theta (I + \alpha u(t))\rho(\theta, t)) = \rho_0(\theta)\delta(t)$$

$$\dot{u} + \beta u = F(I + \alpha u) \qquad F(x) = \frac{\beta}{2\pi} x$$

If $\rho_0 = \frac{1}{2\pi}$

Activity equation

$$\dot{u} + \beta u = \beta(I + \alpha u)\rho(\pi, t)$$

$$\partial_t \rho + \partial_\theta (I + \alpha u(t))\rho(\theta, t)) = \rho_0(\theta)\delta(t)$$

$$\dot{u} + \beta u = F(I + \alpha u)$$

$$F(x) = \frac{\beta}{2\pi} x$$

Wilson-Cowan equation

If $\rho_0 = \frac{1}{2\pi}$

Beyond mean field theory

Need a scheme to compute moments of η

Density functional

e.g. Buice and Chow, PRE, 76.031118, 2007

Density functional

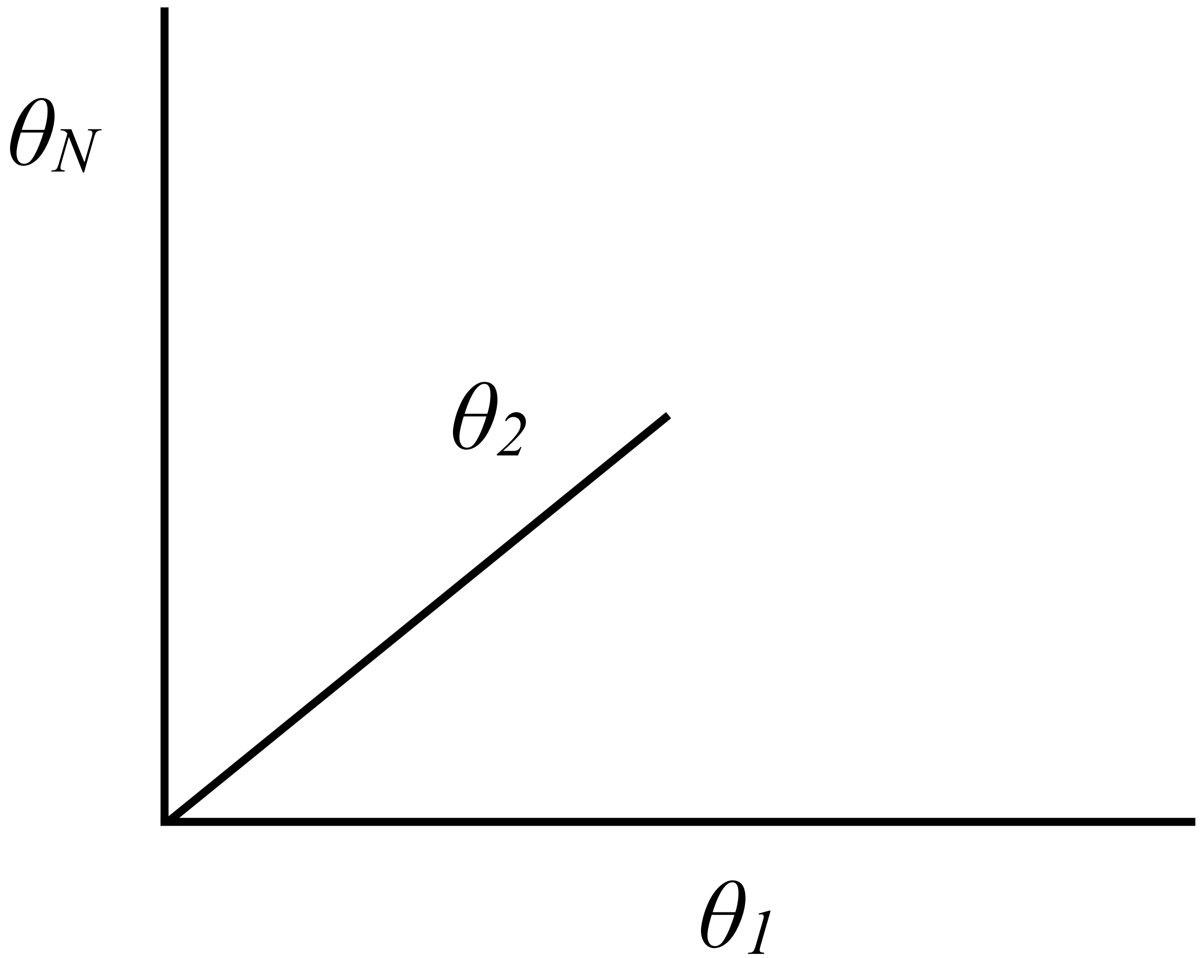
e.g. Buice and Chow, PRE, 76.031118, 2007

Liouville

Density functional

e.g. Buice and Chow, PRE, 76.031118, 2007

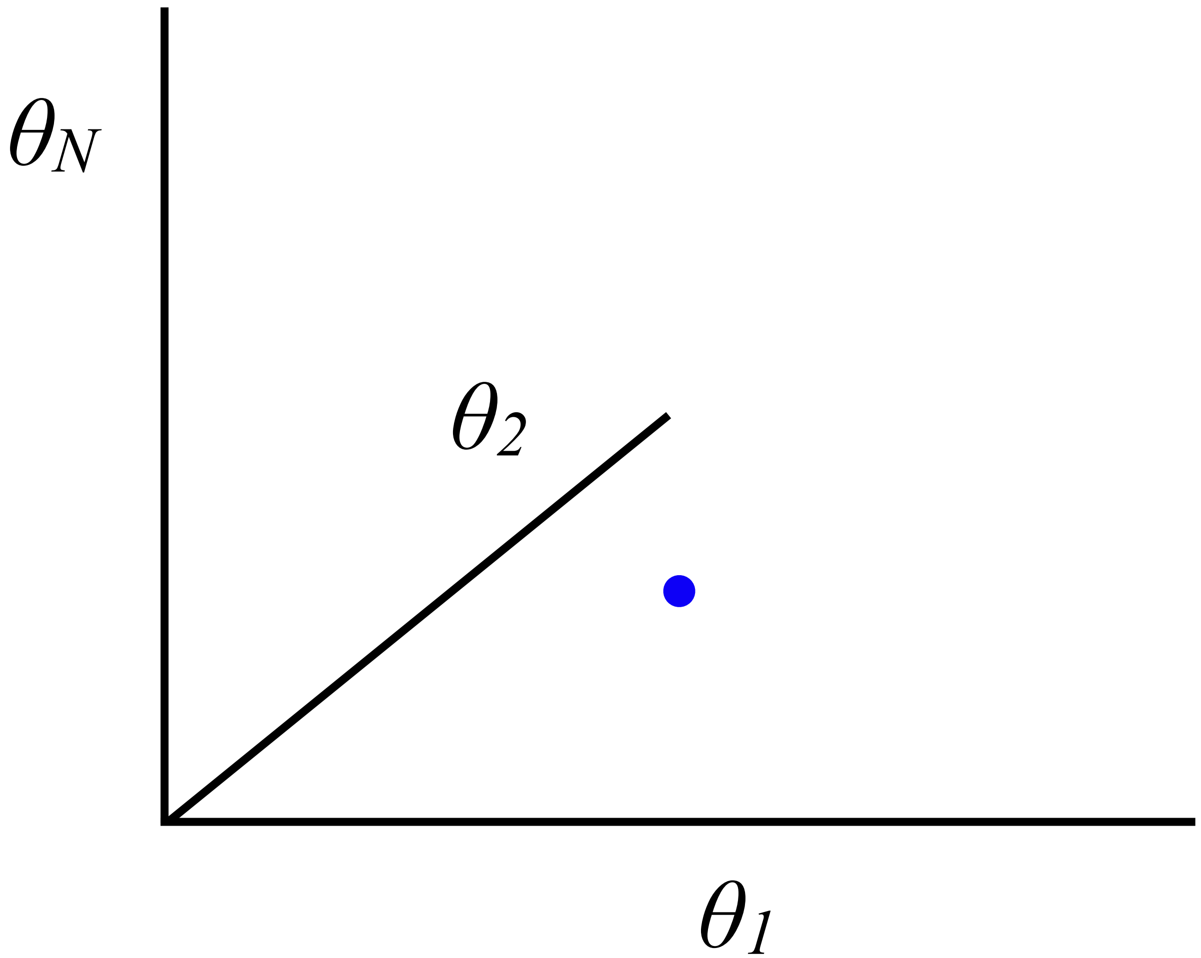
Liouville



Density functional

e.g. Buice and Chow, PRE, 76.031118, 2007

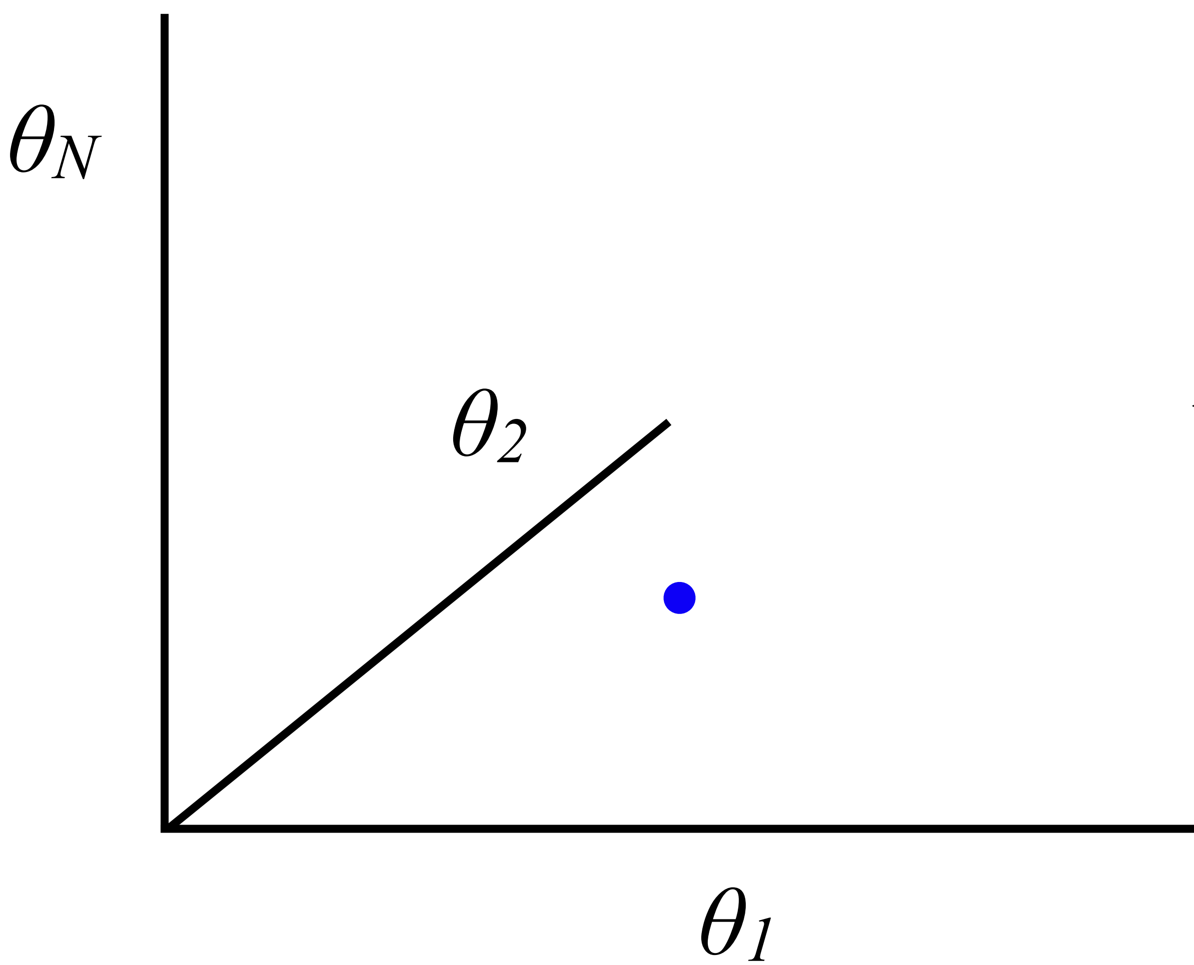
Liouville



Density functional

e.g. Buice and Chow, PRE, 76.031118, 2007

Liouville

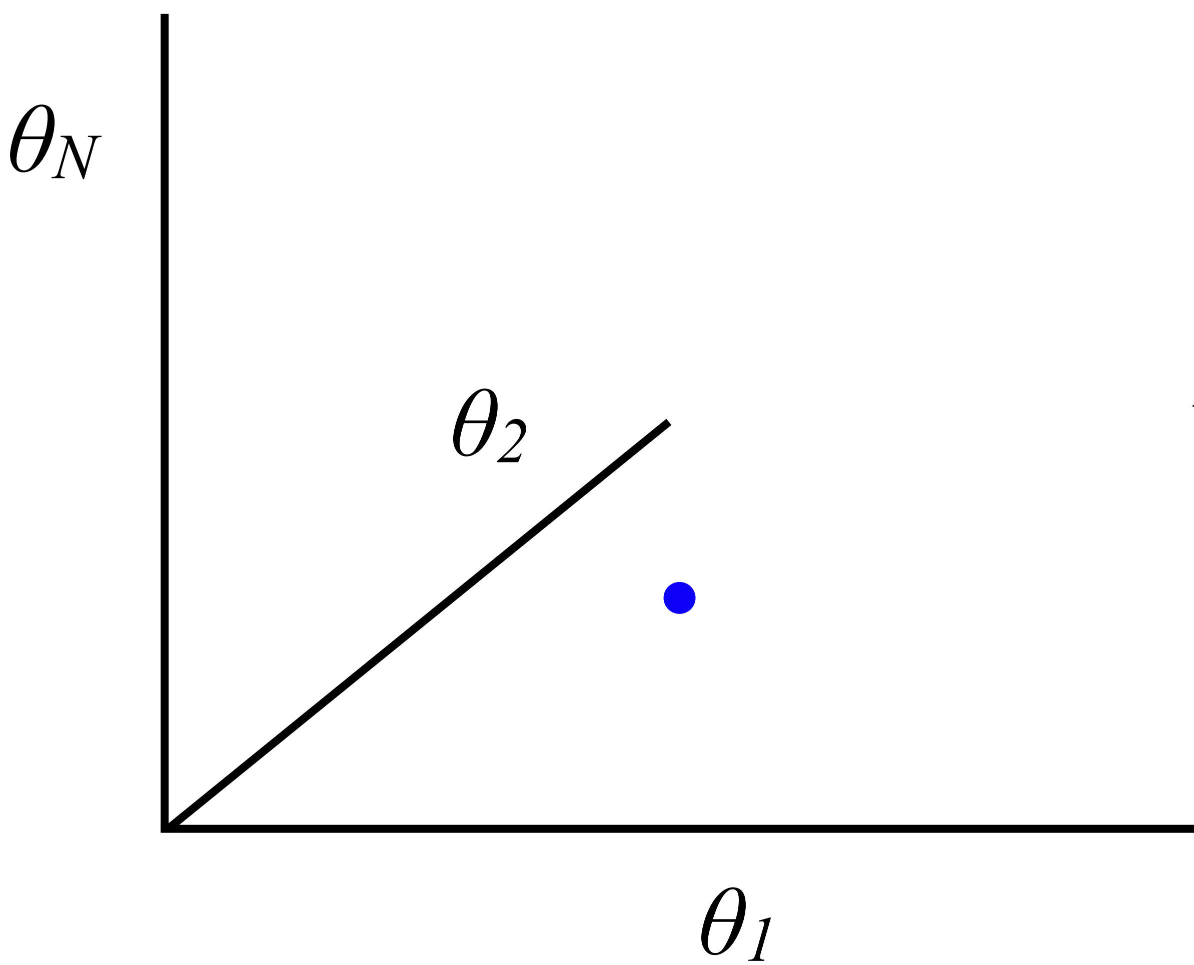


$\longleftrightarrow$

Density functional

e.g. Buice and Chow, PRE, 76.031118, 2007

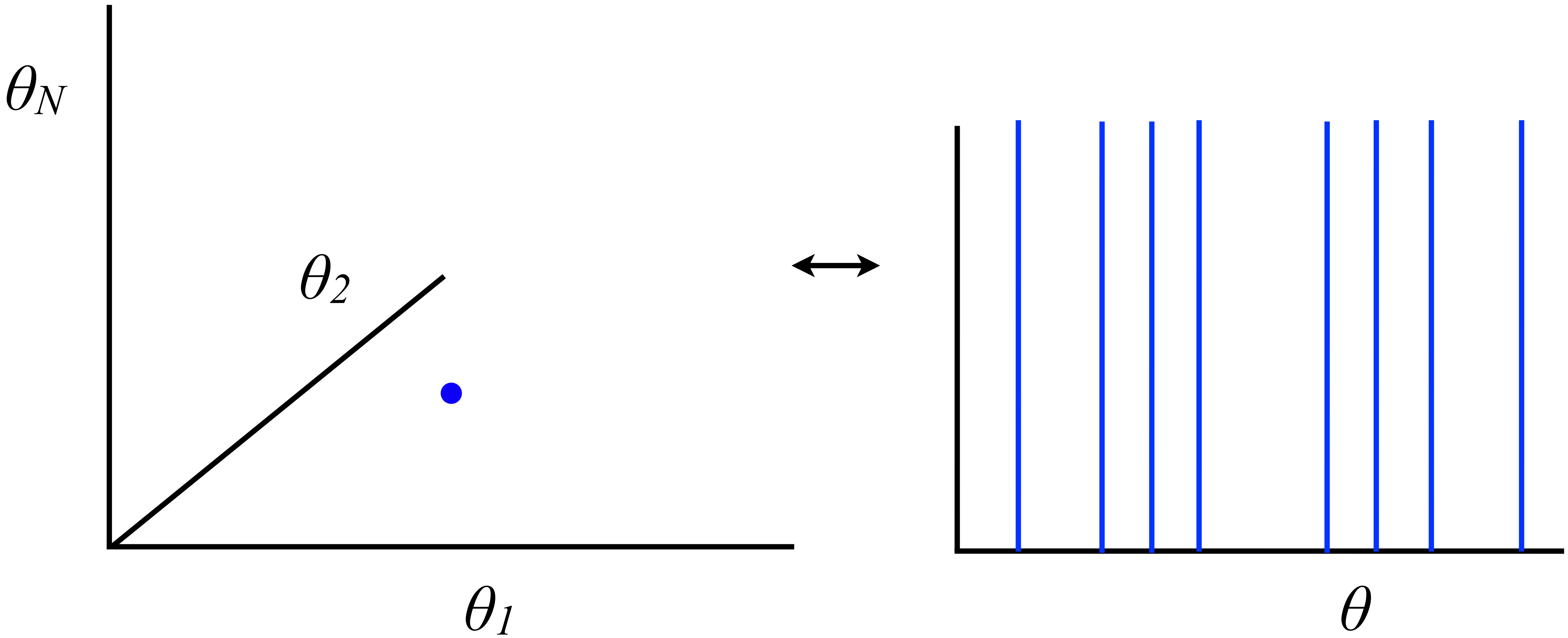
Liouville $\leftrightarrow$ Klimontovich



Density functional

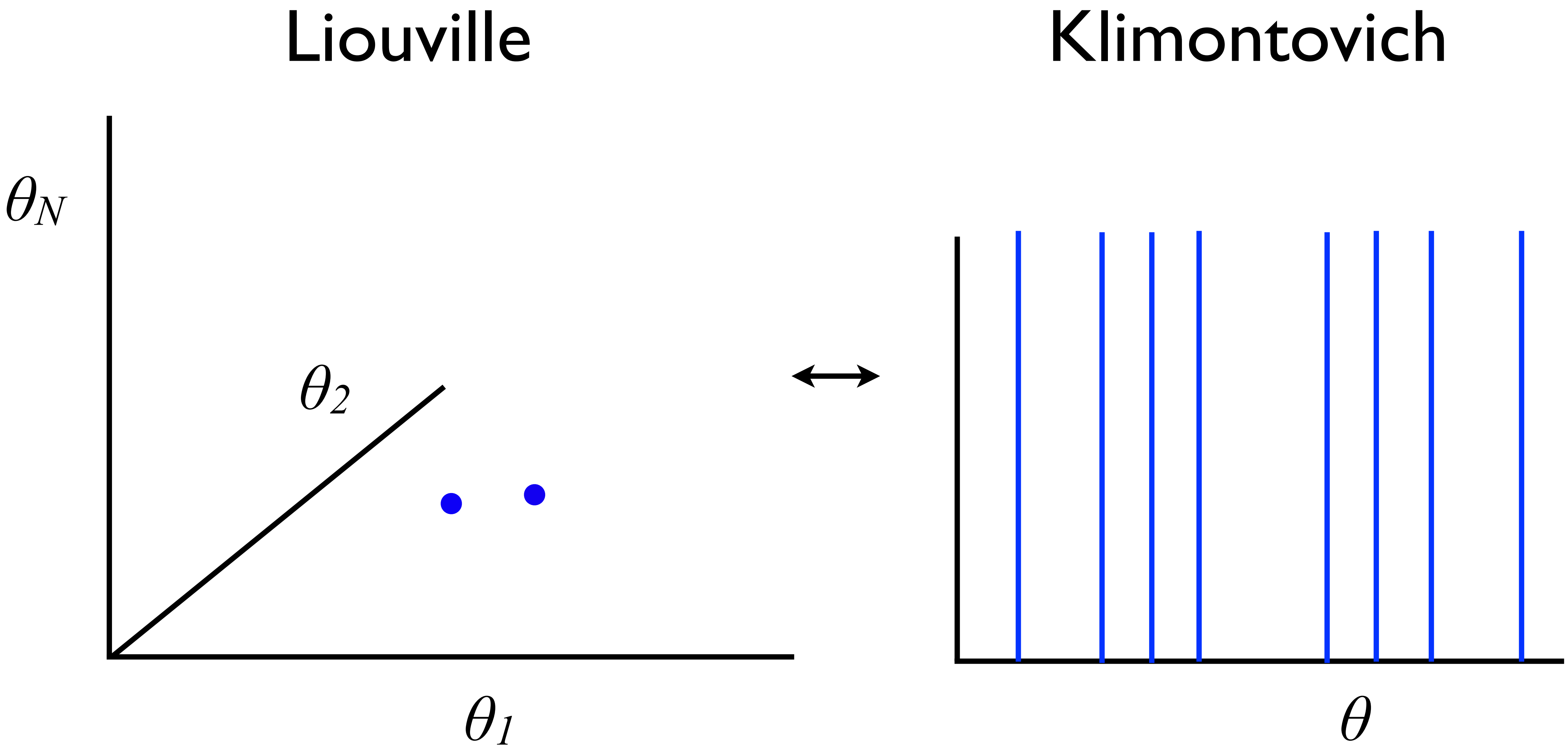
e.g. Buice and Chow, PRE, 76.031118, 2007

Liouville | Klimontovich



Density functional

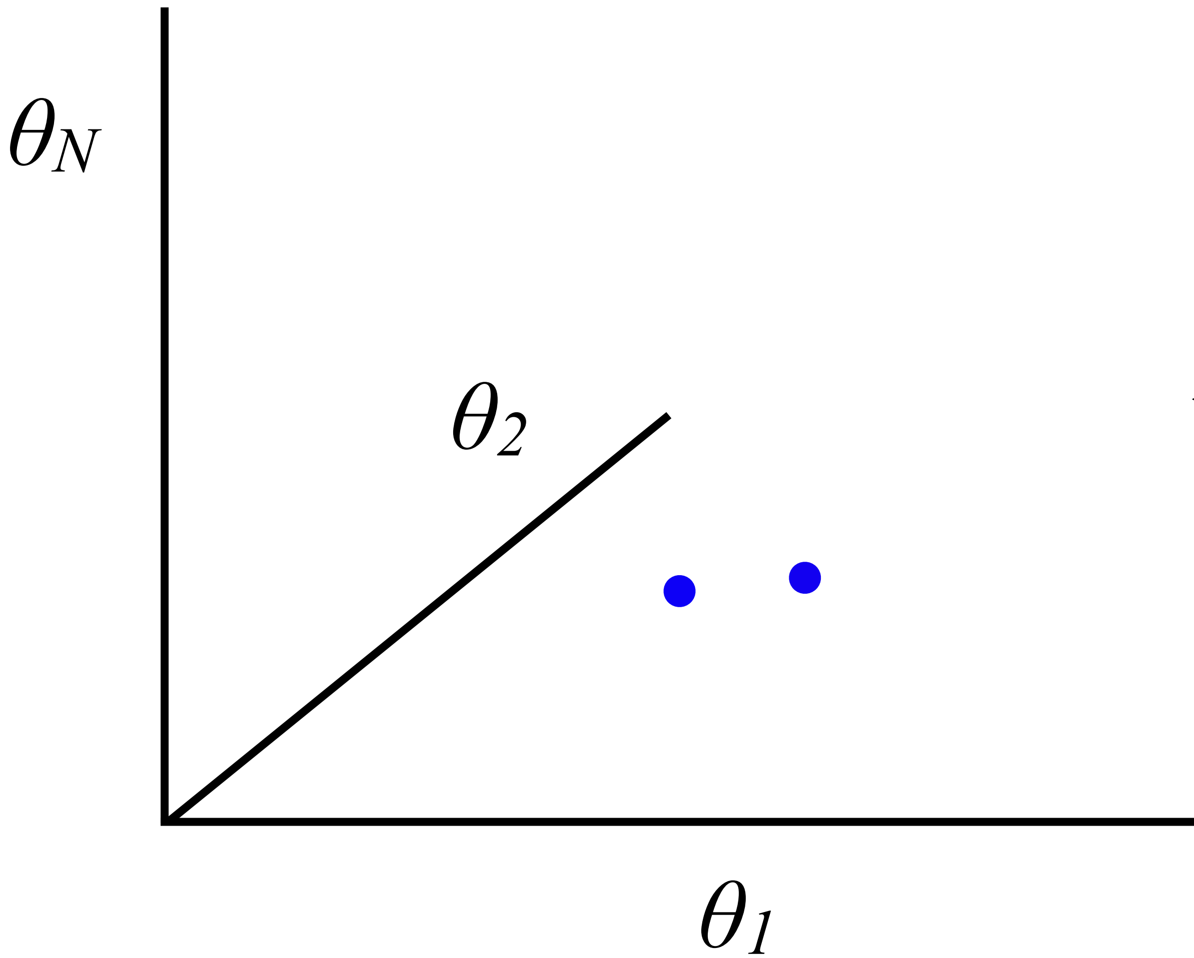
e.g. Buice and Chow, PRE, 76.031118, 2007



Density functional

e.g. Buice and Chow, PRE, 76.031118, 2007

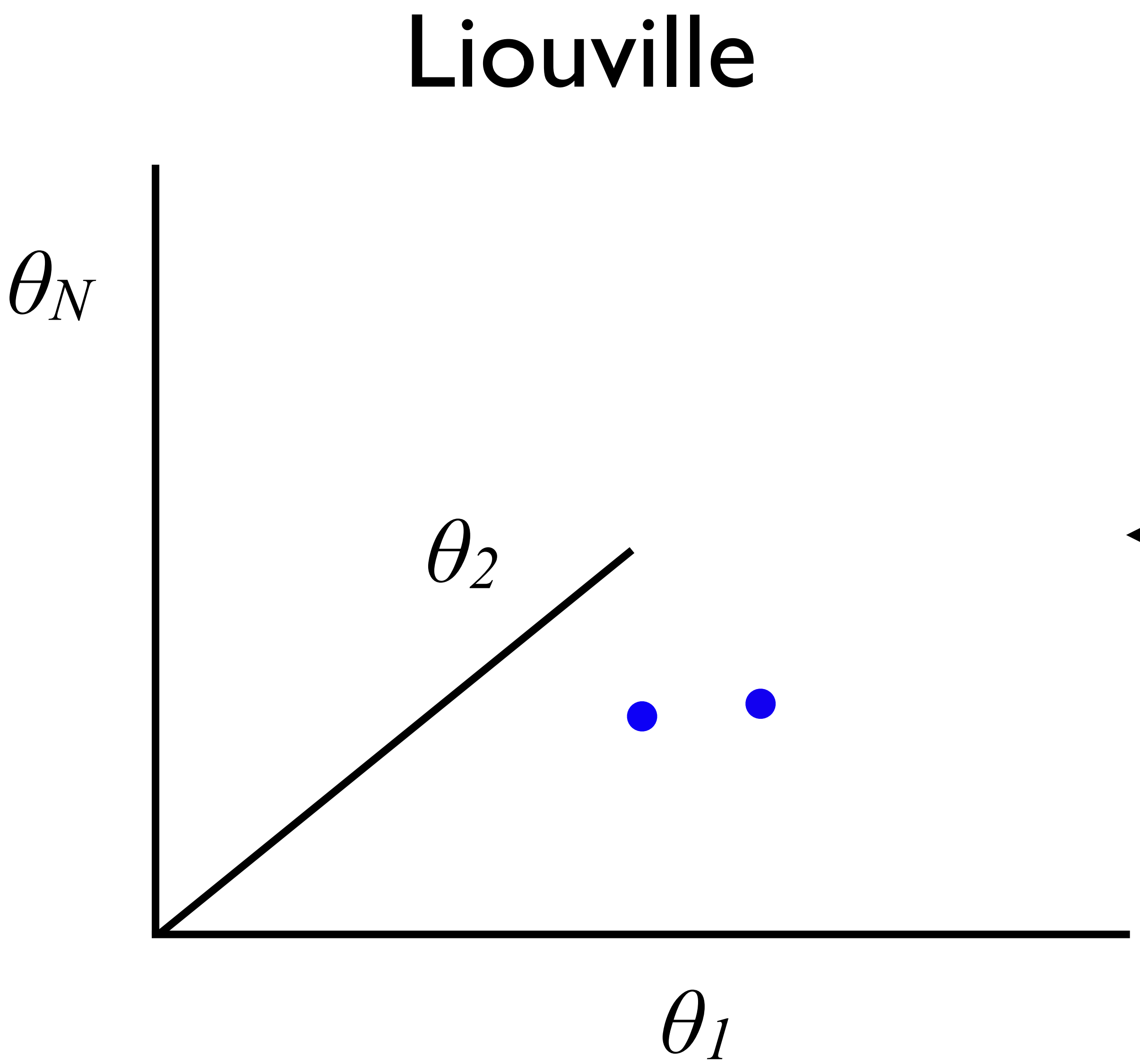
Liouville $\leftrightarrow$ Klimontovich



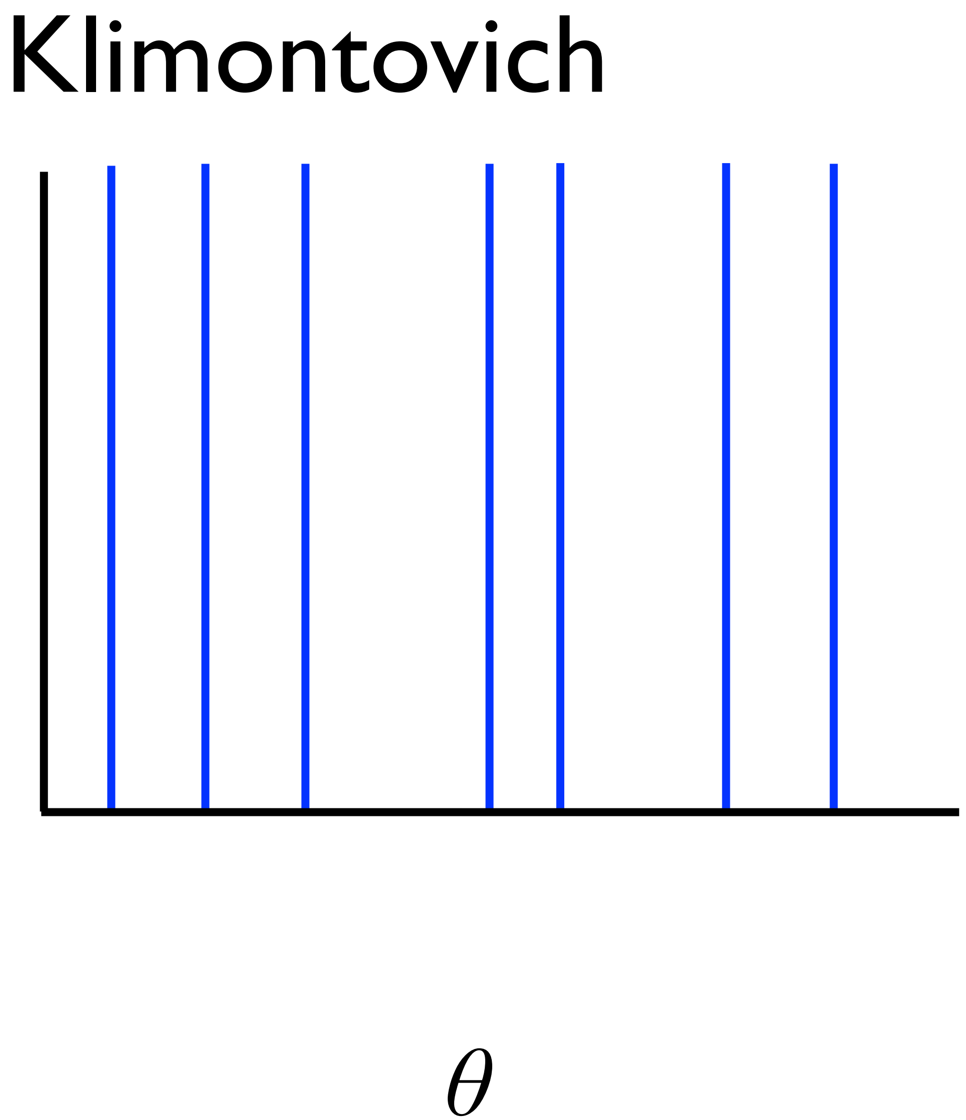
θ

Density functional

e.g. Buice and Chow, PRE, 76.031118, 2007



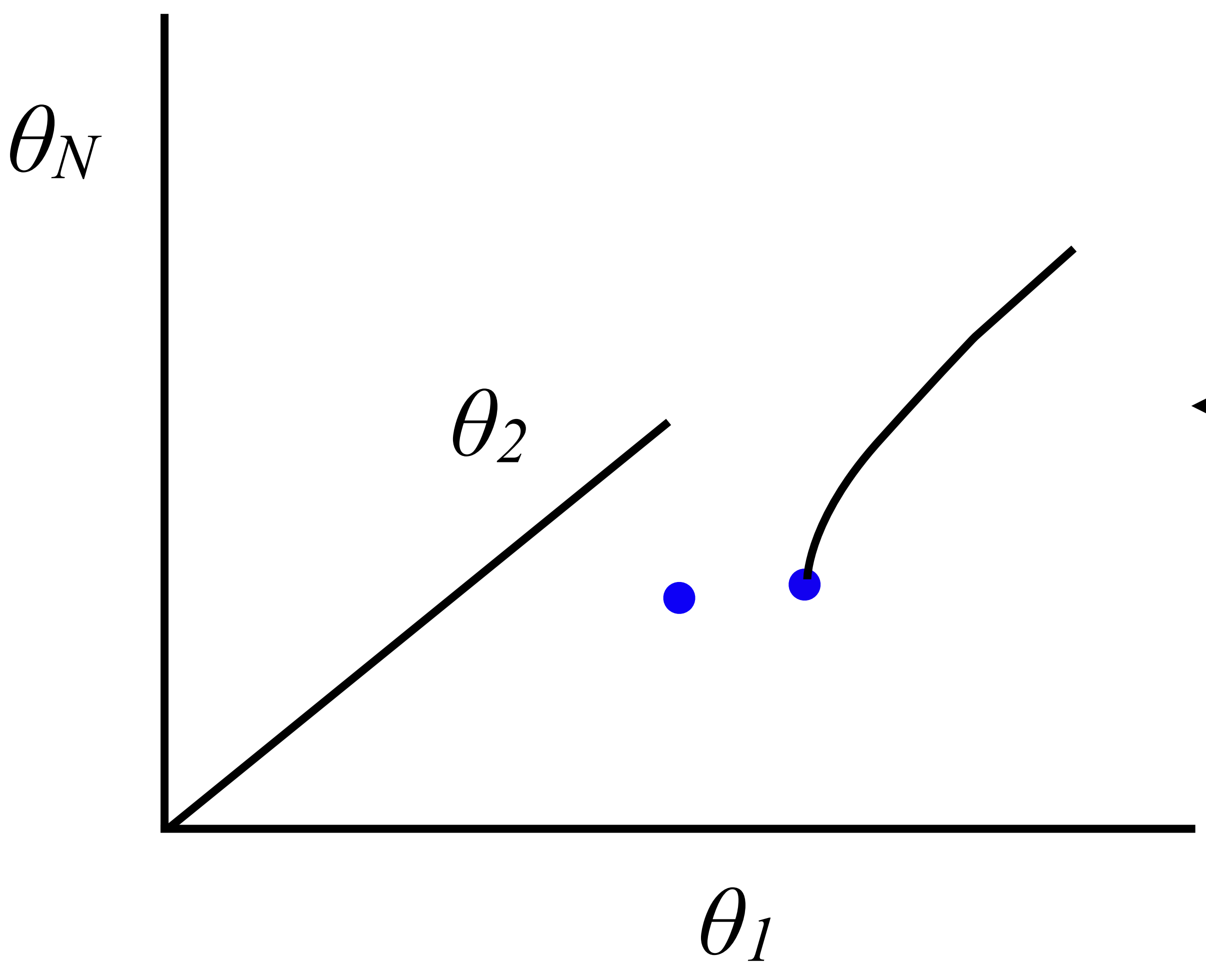
$\leftrightarrow$

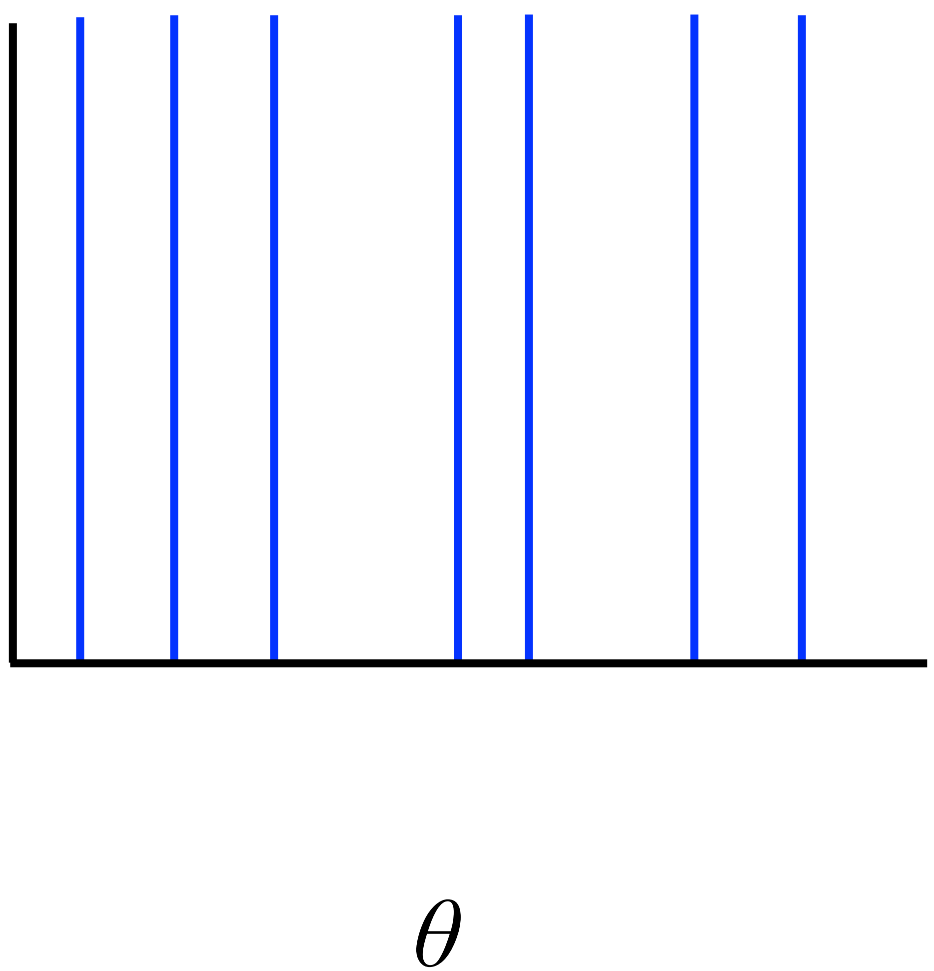


Density functional

e.g. Buice and Chow, PRE, 76.031118, 2007

Liouville ↔ Klimontovich

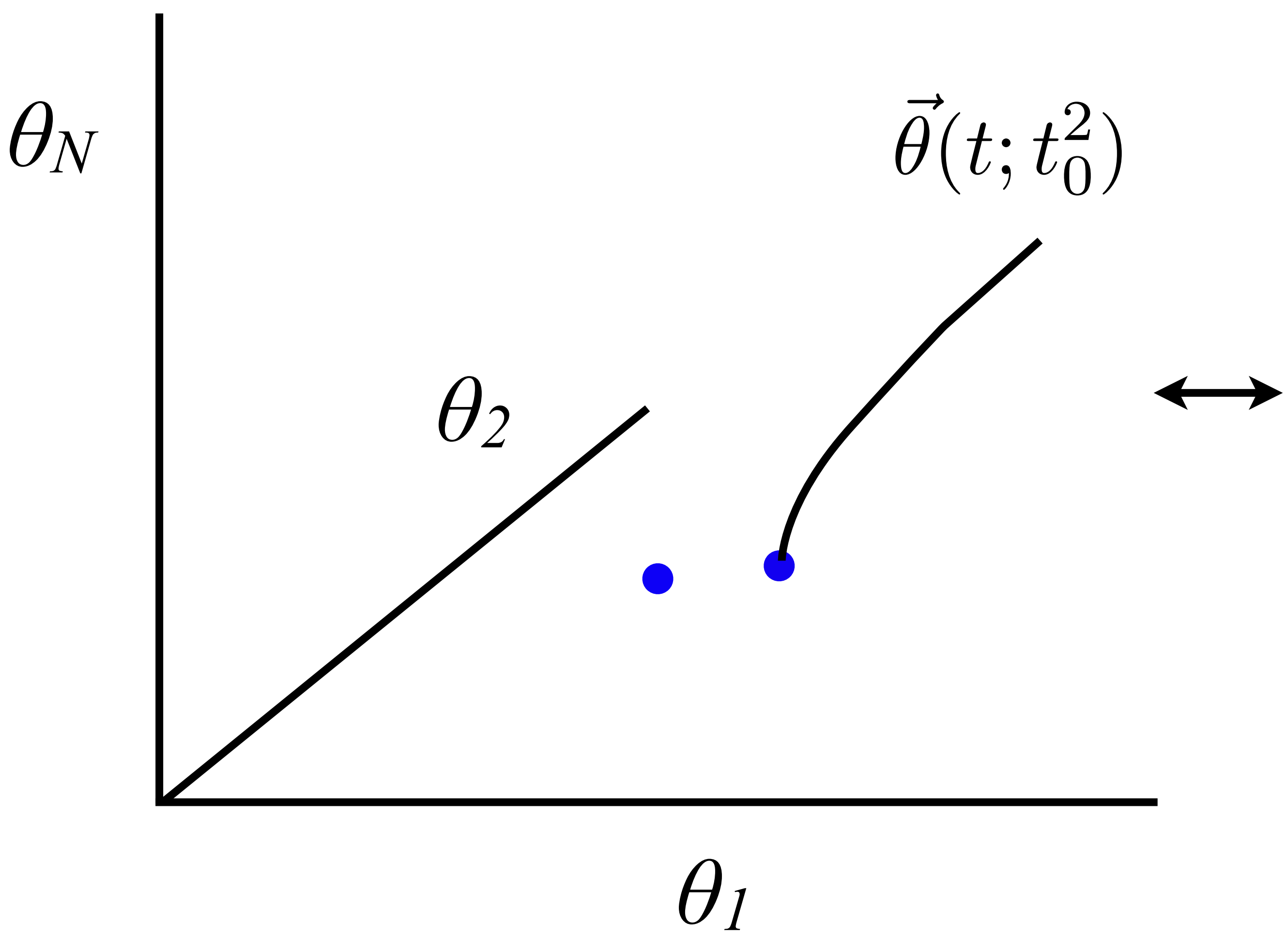


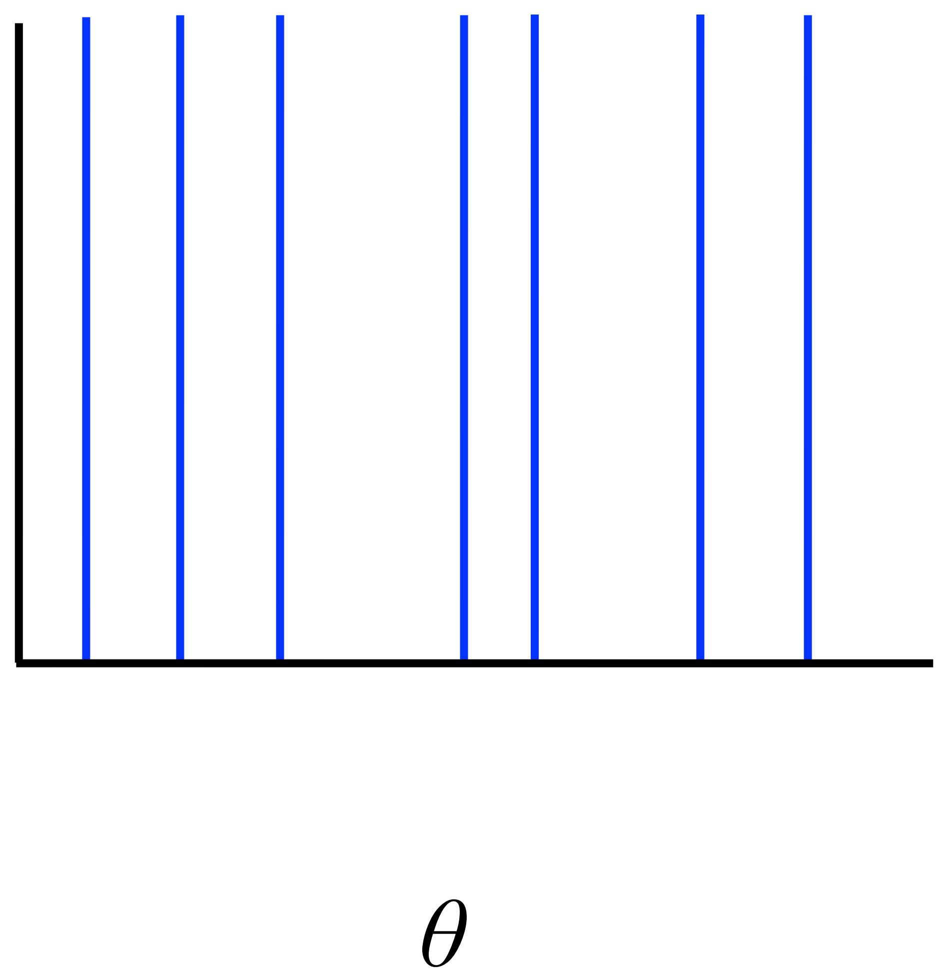


Density functional

e.g. Buice and Chow, PRE, 76.031118, 2007

Liouville ↔ Klimontovich

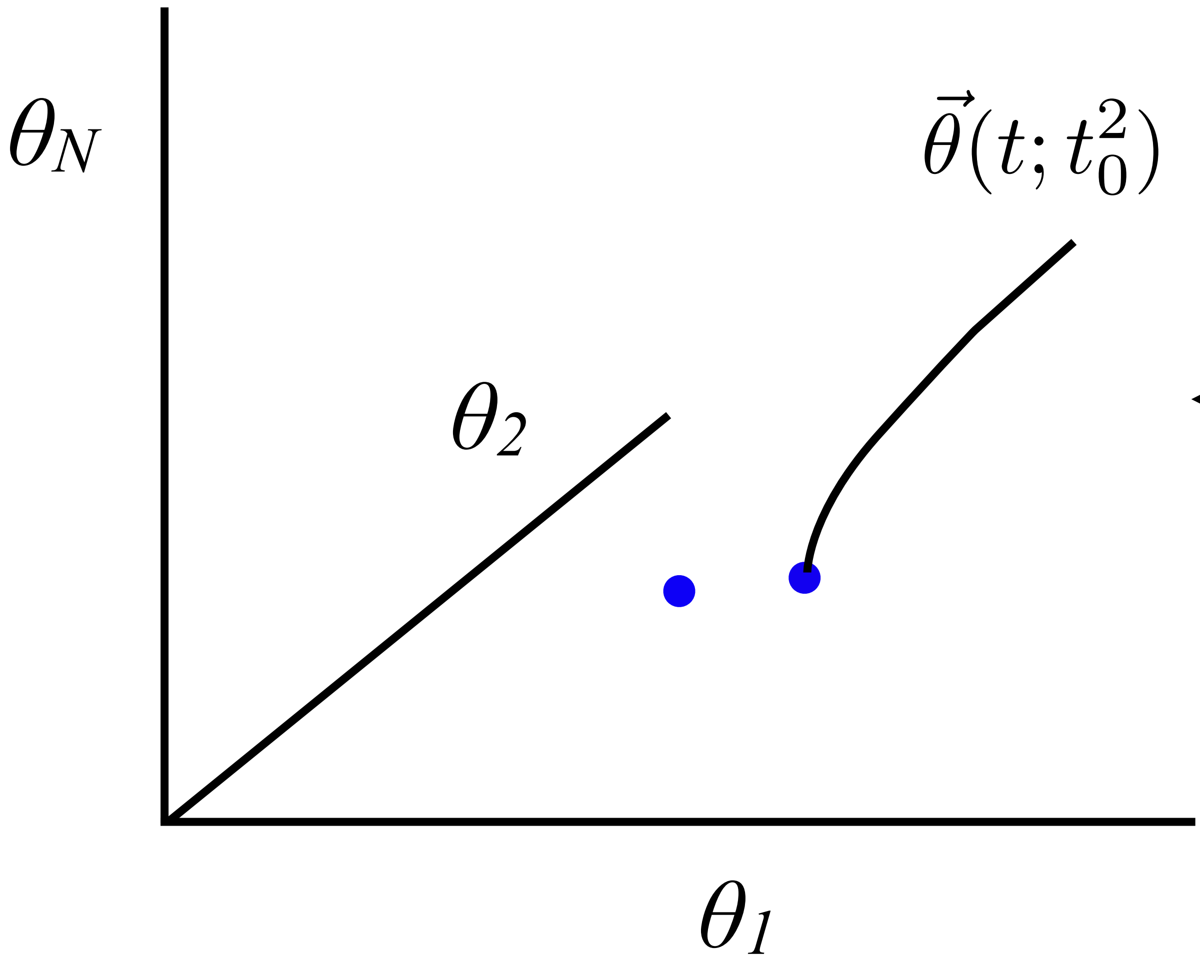




Density functional

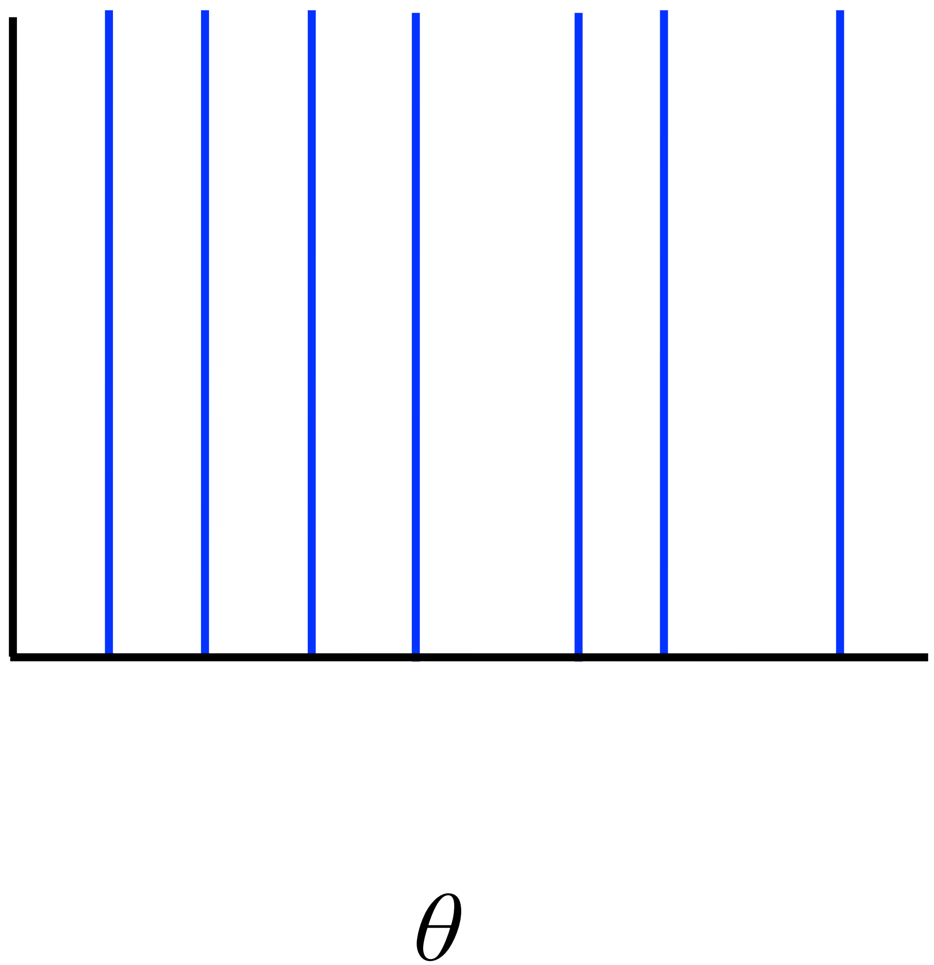
e.g. Buice and Chow, PRE, 76.031118, 2007

Liouville



$\leftrightarrow$

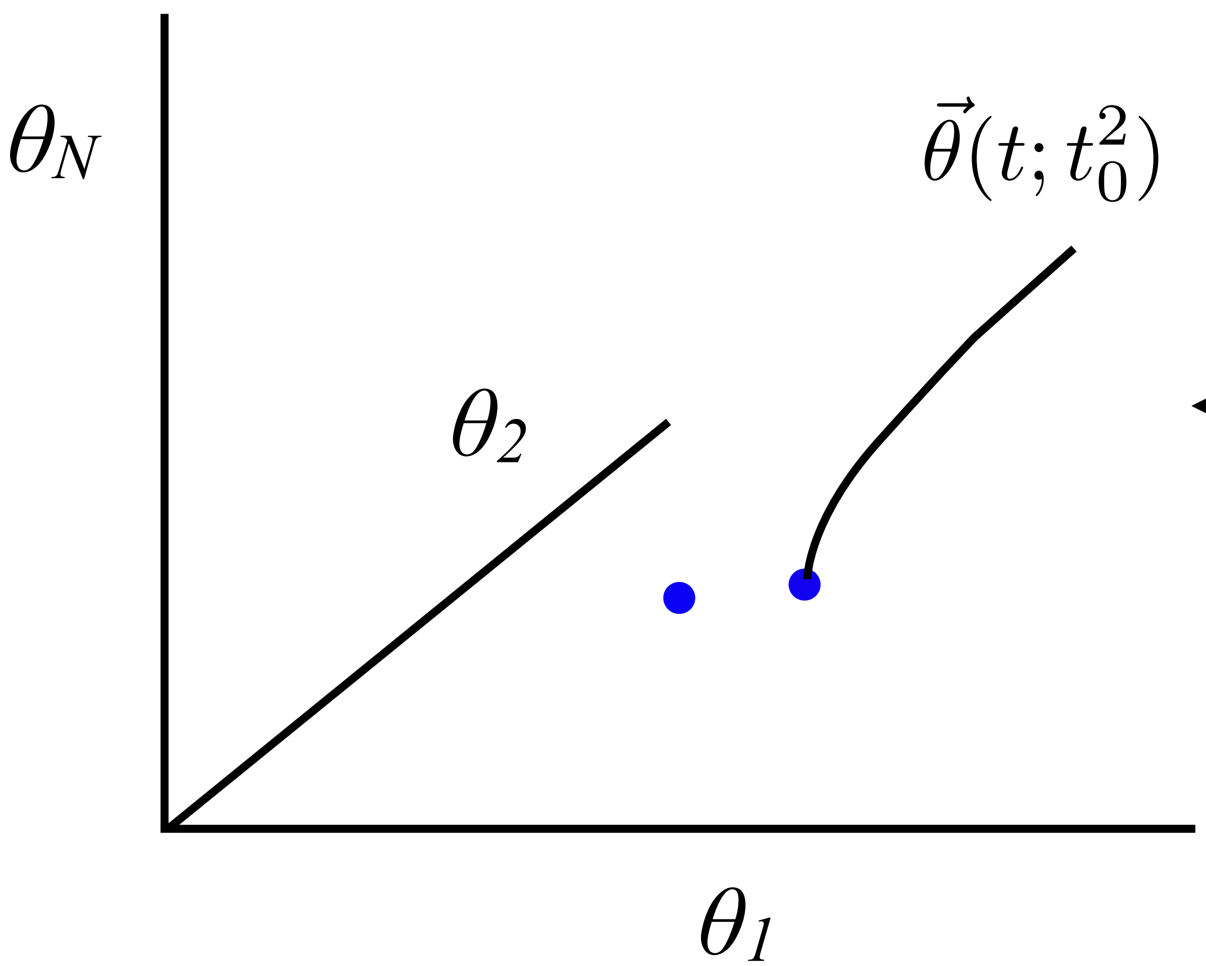
Klimontovich



Density functional

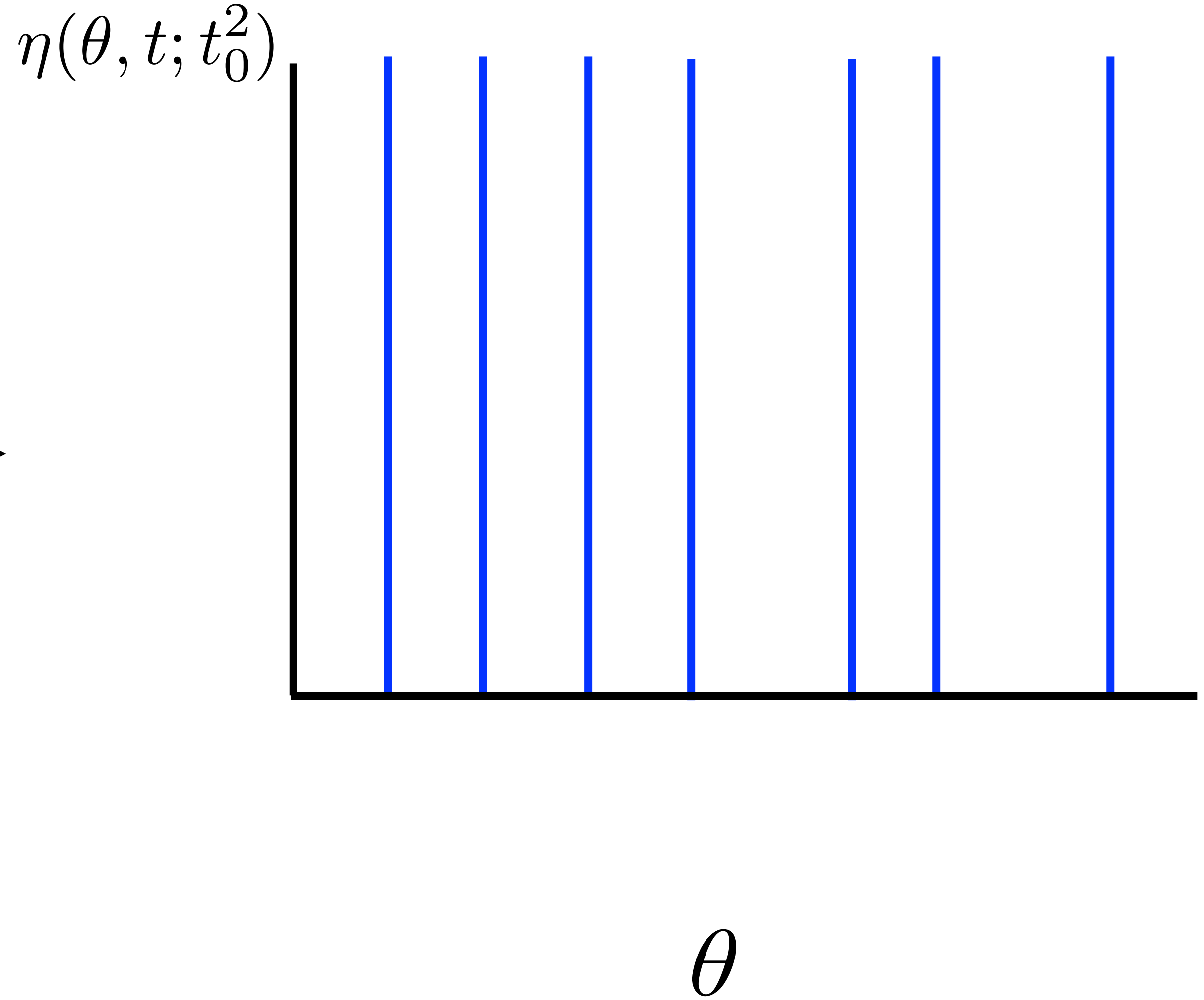
e.g. Buice and Chow, PRE, 76.031118, 2007

Liouville



$\leftrightarrow$

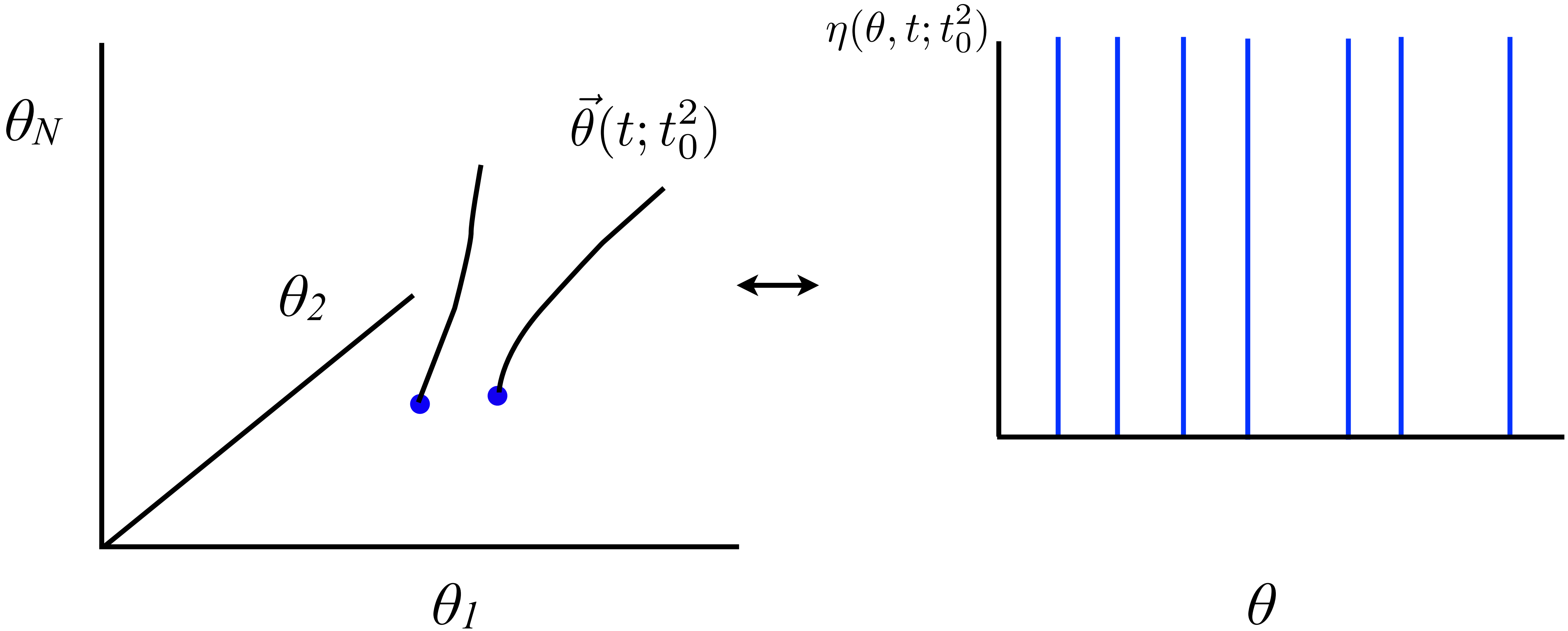
Klimontovich



Density functional

e.g. Buice and Chow, PRE, 76.031118, 2007

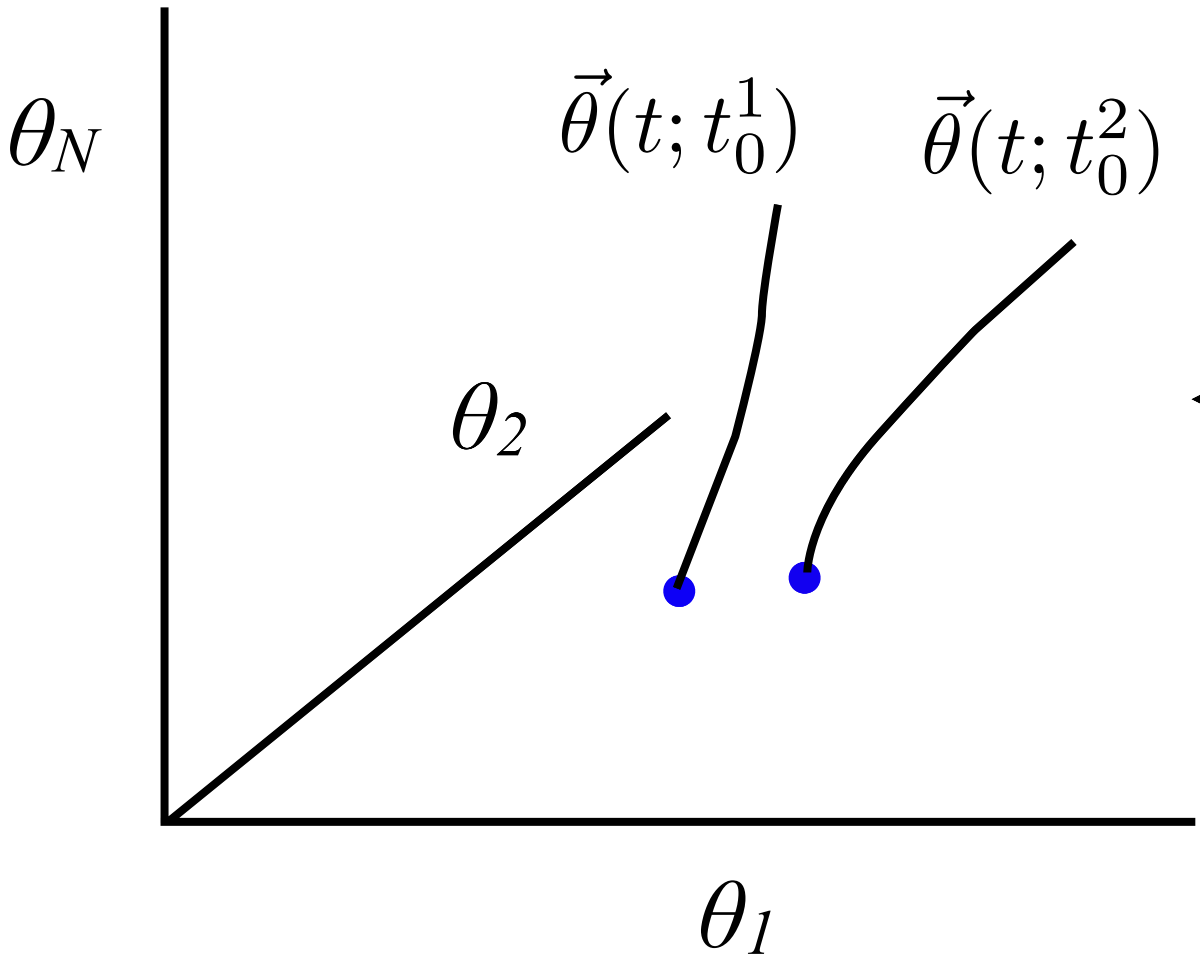
Liouville | Klimontovich



Density functional

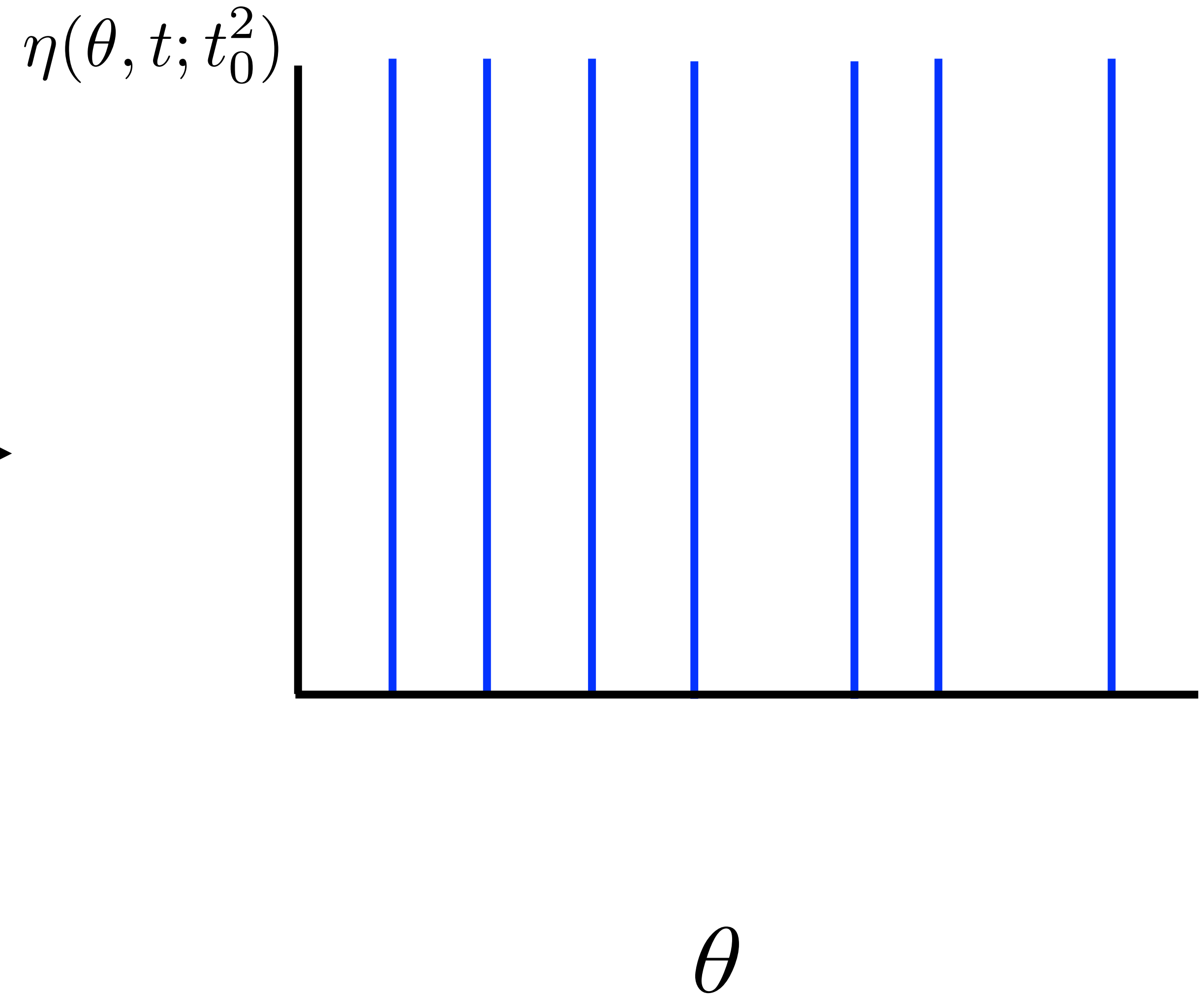
e.g. Buice and Chow, PRE, 76.031118, 2007

Liouville



$\leftrightarrow$

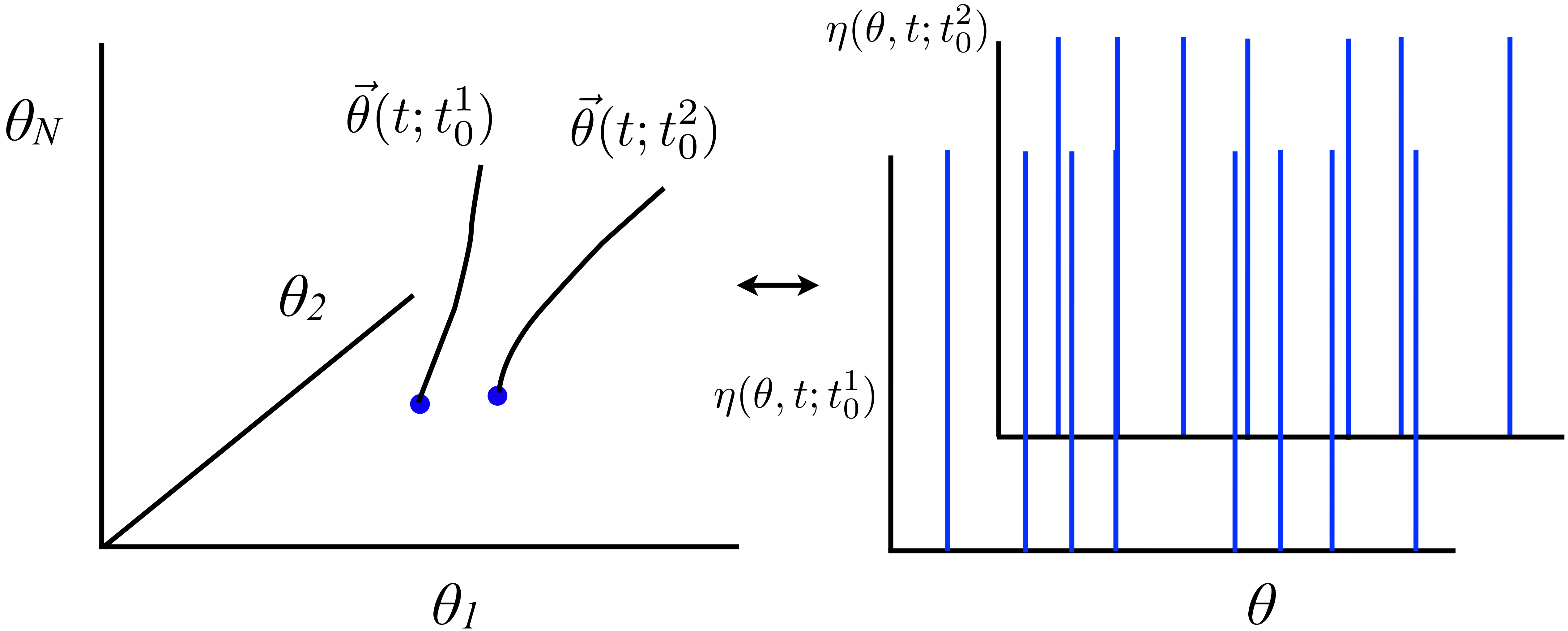
Klimontovich



Density functional

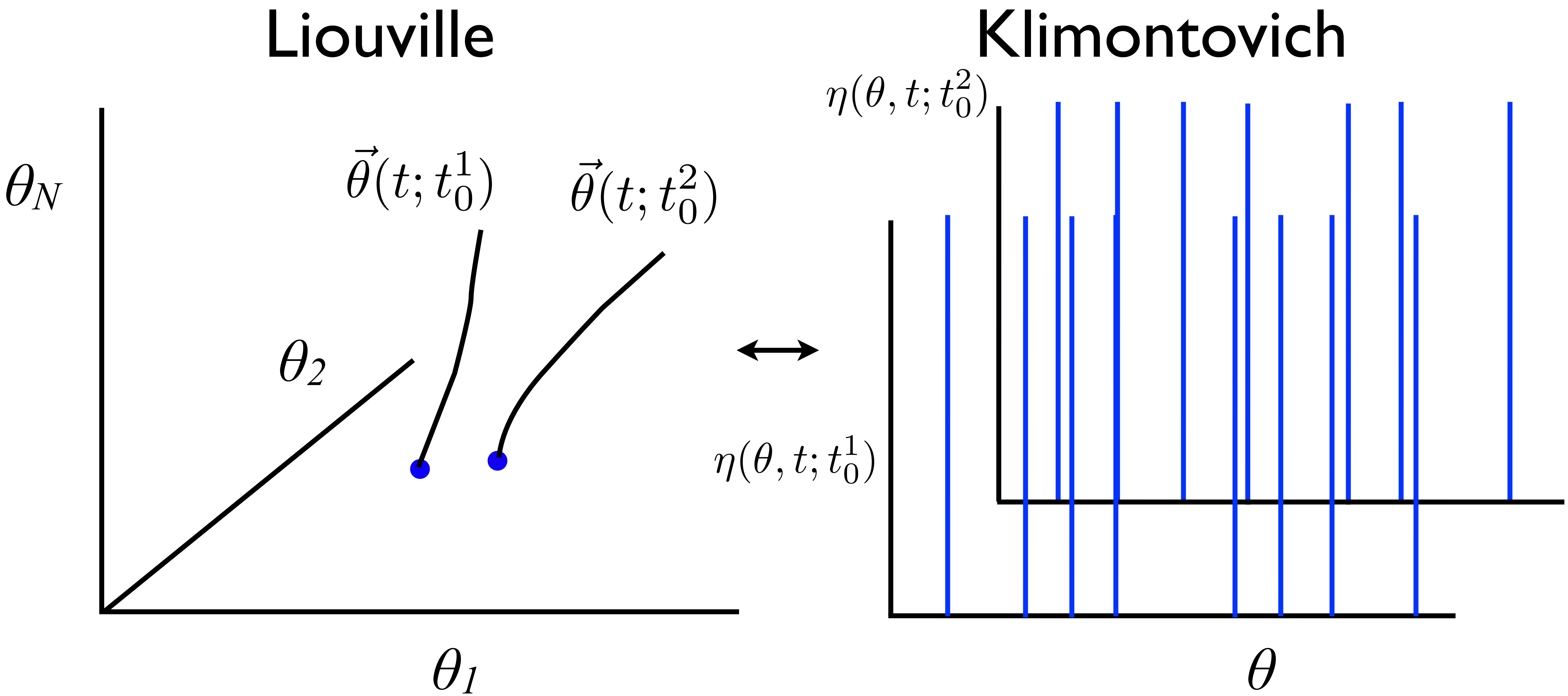
e.g. Buice and Chow, PRE, 76.031118, 2007

Liouville Klimontovich



Density functional

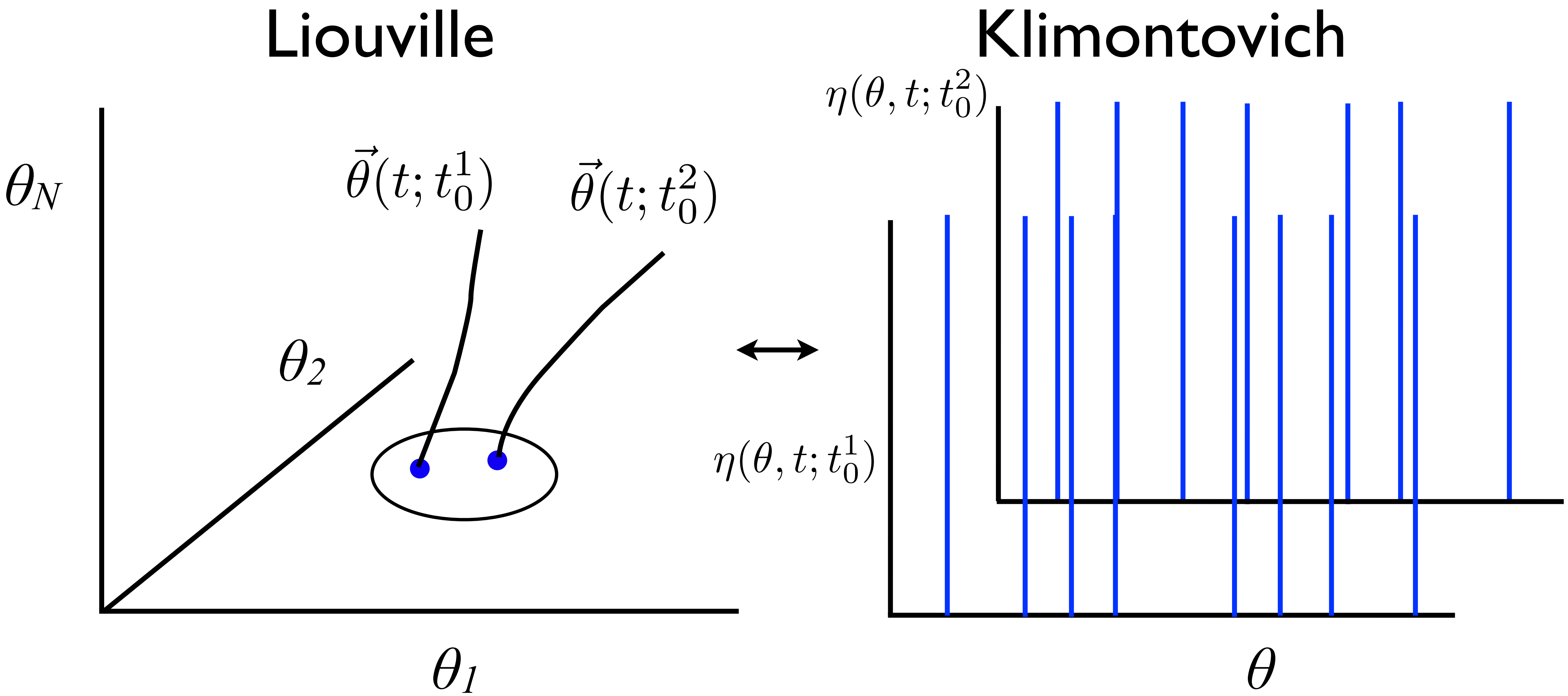
e.g. Buice and Chow, PRE, 76.031118, 2007



Ensemble of initial data

Density functional

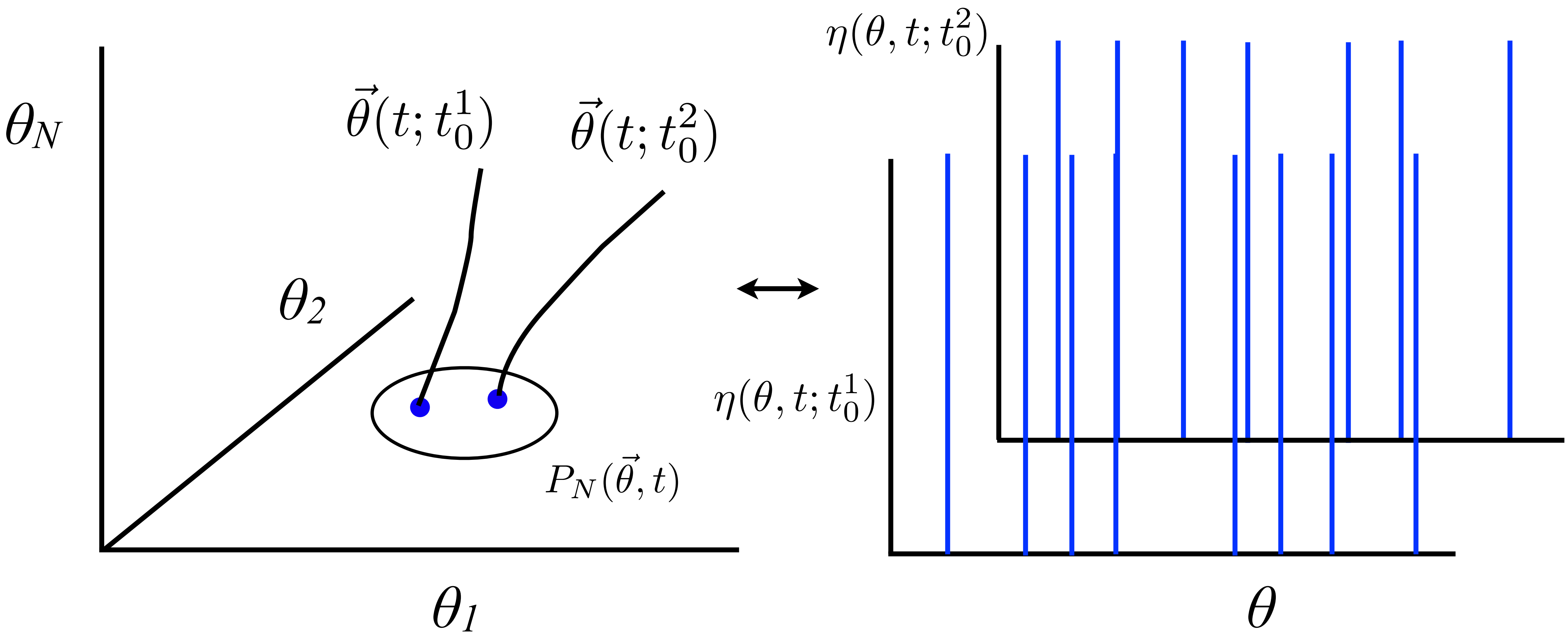
e.g. Buice and Chow, PRE, 76.031118, 2007



Ensemble of initial data

Density functional

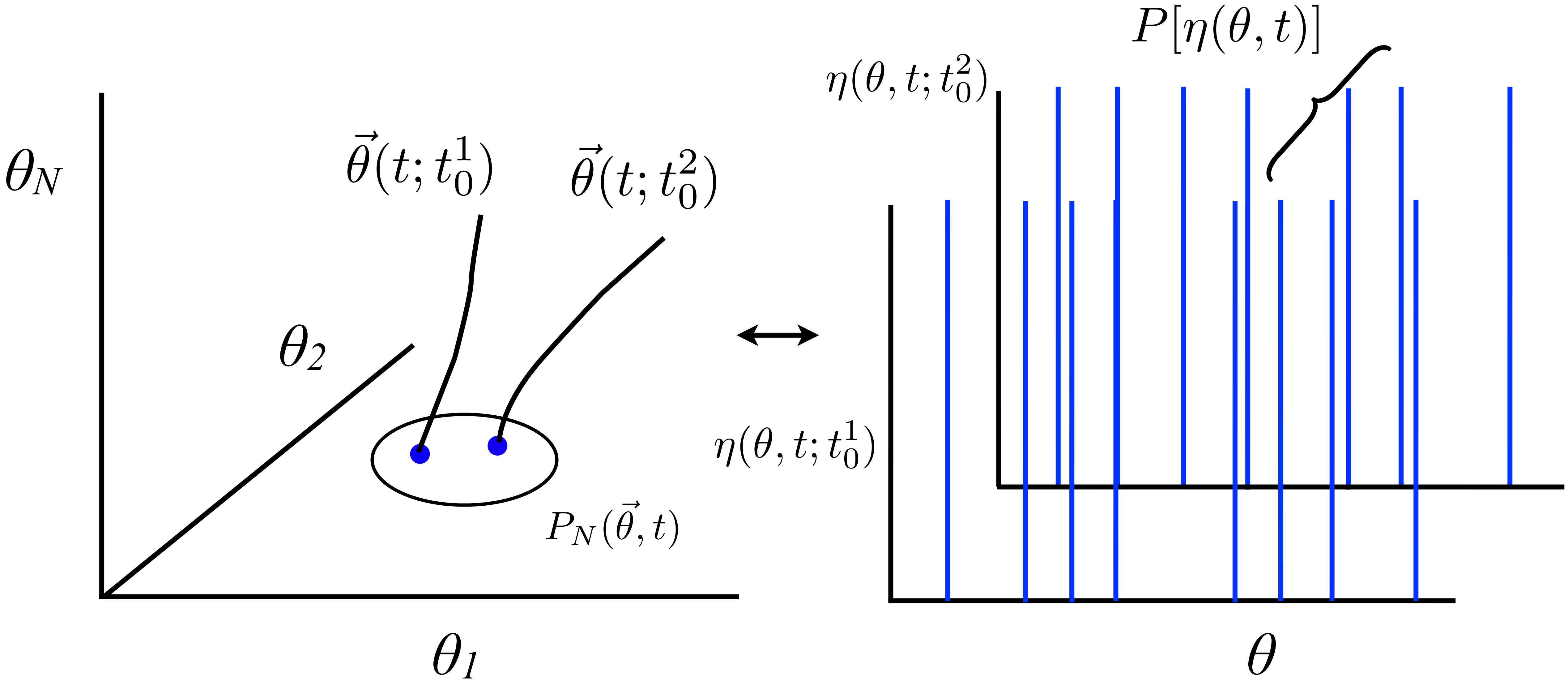
e.g. Buice and Chow, PRE, 76.031118, 2007



Ensemble of initial data $\Rightarrow$ Ensemble of systems

Density functional

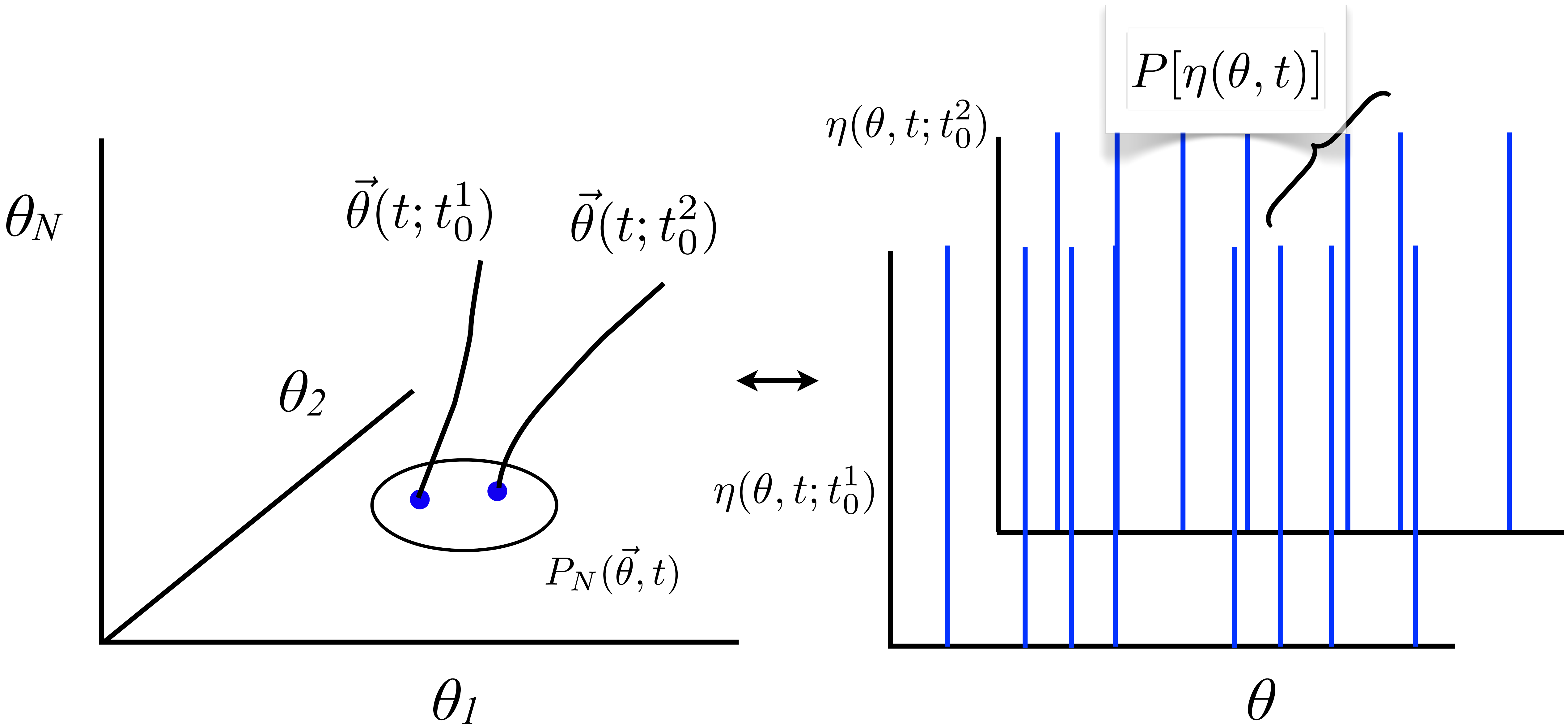
e.g. Buice and Chow, PRE, 76.031118, 2007



Ensemble of initial data

Density functional

e.g. Buice and Chow, PRE, 76.031118, 2007



Ensemble of initial data $\Rightarrow$ Density of densities

Density functional

e.g. Buice and Chow, PRE, 76.031118, 2007

$$\partial_t \eta + \partial_\theta \left[(I(t) + \alpha u(t)) \eta \right] = 0$$

$$\dot{u} + \beta u - \beta (I + \alpha u) \eta(\pi, t) = 0$$

Density functional

e.g. Buice and Chow, PRE, 76.031118, 2007

$$\partial_t \eta + \partial_\theta \left[(I(t) + \alpha u(t))\eta \right] = 0$$

$$\dot{u} + \beta u - \beta(I + \alpha u)\eta(\pi, t) = 0$$

$$\eta(\theta, t_0) = \eta_0(\theta)$$

Density functional

e.g. Buice and Chow, PRE, 76.031118, 2007

$$\partial_t \eta + \partial_\theta \left[(I(t) + \alpha u(t))\eta \right] = 0$$

$$\dot{u} + \beta u - \beta(I + \alpha u)\eta(\pi, t) = 0$$

$$\eta(\theta, t_0) = \eta_0(\theta) \quad u(t_0) = u_0$$

Density functional

e.g. Buice and Chow, PRE, 76.031118, 2007

$$\left.\begin{aligned} &\partial_t \eta + \partial_\theta \left[(I(t) + \alpha u(t))\eta \right] = 0 \\ &\dot{u} + \beta u - \beta(I + \alpha u)\eta(\pi, t) = 0 \\ &\eta(\theta, t_0) = \eta_0(\theta) \quad u(t_0) = u_0 \end{aligned}\right\}$$

Density functional

e.g. Buice and Chow, PRE, 76.031118, 2007

$$\left.\begin{aligned} &\partial_t \eta + \partial_\theta \left[(I(t) + \alpha u(t))\eta \right] = 0 \\ &\dot{u} + \beta u - \beta (I + \alpha u) \eta(\pi, t) = 0 \\ &\eta(\theta, t_0) = \eta_0(\theta) \quad u(t_0) = u_0 \end{aligned}\right\} \qquad \mathcal{L}(u, \eta | u_0, \eta_0) = 0$$

Density functional

e.g. Buice and Chow, PRE, 76.031118, 2007

$$\left.\begin{aligned}
&\partial_t \eta + \partial_\theta \left[(I(t) + \alpha u(t)) \eta \right] = 0 \\
&\dot{u} + \beta u - \beta (I + \alpha u) \eta(\pi, t) = 0 \\
&\eta(\theta, t_0) = \eta_0(\theta) \quad u(t_0) = u_0
\end{aligned}\right\} \qquad \mathcal{L}(u, \eta | u_0, \eta_0) = 0$$

$$P[u, \eta | u_0, \eta_0] \propto \delta[\mathcal{L}]$$

Density functional

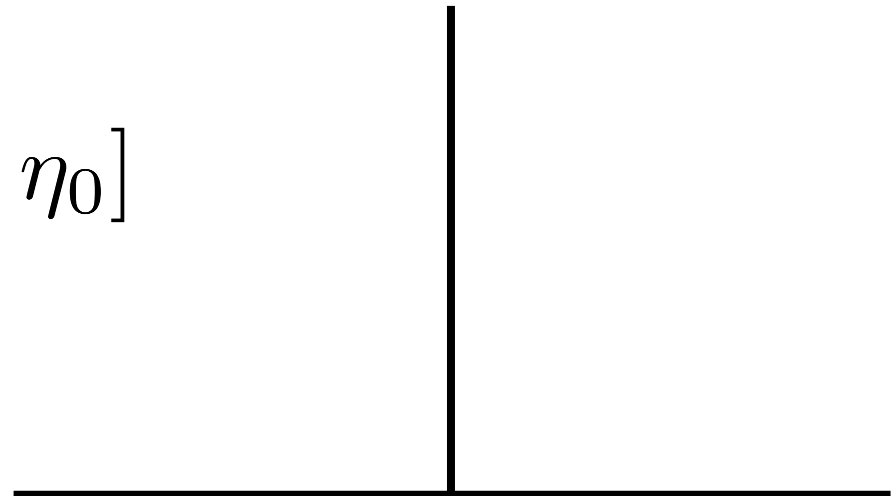
e.g. Buice and Chow, PRE, 76.031118, 2007

$$\left.\begin{array}{l} \partial_t \eta + \partial_\theta \left[(I(t) + \alpha u(t)) \eta \right] = 0 \\ \dot{u} + \beta u - \beta (I + \alpha u) \eta(\pi, t) = 0 \\ \eta(\theta, t_0) = \eta_0(\theta) \quad u(t_0) = u_0 \end{array}\right\} \qquad \mathcal{L}(u, \eta | u_0, \eta_0) = 0$$

$$P[u, \eta | u_0, \eta_0] \propto \delta[\mathcal{L}]$$

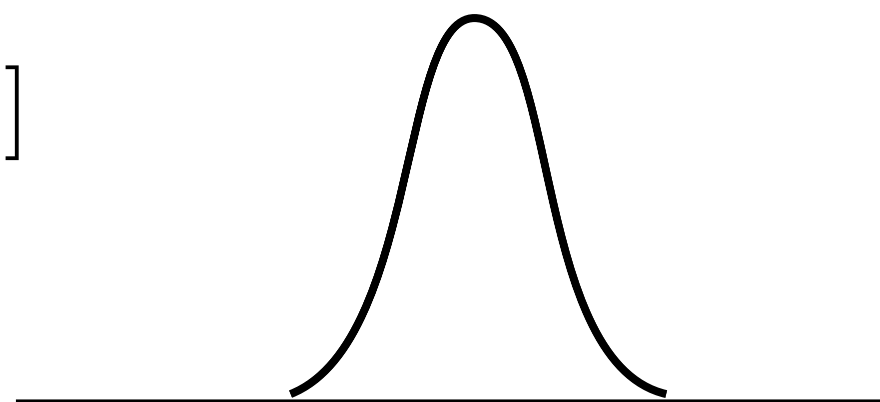
Density of the density

$P[u, \eta | u_0, \eta_0]$



u, η

$P[u_0, \eta_0]$



uncertainty
in initial data

$$P[u, \eta] = \int \mathcal{D}u_0 \mathcal{D}\eta_0 P[u, \eta | u_0, \eta_0] P[u_0, \eta_0]$$

$$\delta(x) = \int e^{ikx} dk$$

$$P[u,\eta] = \delta[\mathcal{L}] \propto \int \mathcal{D}\tilde{u}\mathcal{D}\tilde{\eta}\, e^{-S[u,\tilde{u},\eta,\tilde{\eta}]}$$

$$\delta(x) = \int e^{ikx} dk$$

Action

$$P[u,\eta] = \delta[\mathcal{L}] \propto \int \mathcal{D}\tilde{u}\mathcal{D}\tilde{\eta}\, e^{-S[u,\tilde{u},\eta,\tilde{\eta}]}$$

$$\delta(x) = \int e^{ikx} dk$$

Action

$$P[u,\eta] = \delta[\mathcal{L}] \propto \int \mathcal{D}\tilde{u}\mathcal{D}\tilde{\eta}\, e^{-S[u,\tilde{u},\eta,\tilde{\eta}]}$$

Path or functional integral

$$\delta(x) = \int e^{ikx} dk$$

Action

$$P[u,\eta] = \delta[\mathcal{L}] \propto \int \mathcal{D}\tilde{u}\mathcal{D}\tilde{\eta}\, e^{-S[u,\tilde{u},\eta,\tilde{\eta}]}$$

Path or functional integral

$$S[u,\tilde{u},\eta,\tilde{\eta}] = N \int dt d\theta\, \tilde{\eta}(\theta,t)\left(\partial_t \eta + \partial_\theta[(I+\alpha u)\eta]\right)$$

$$+ \int dt\, \tilde{u}\left(\dot{u} + \beta u - \beta[I+\alpha u]\eta(\pi,t)\right)$$

$$\delta(x) = \int e^{ikx} dk$$

Action

$$P[u,\eta] = \delta[\mathcal{L}] \propto \int \mathcal{D}\tilde{u}\mathcal{D}\tilde{\eta}\, e^{-S[u,\tilde{u},\eta,\tilde{\eta}]}$$

Path or functional integral

$$S[u,\tilde{u},\eta,\tilde{\eta}] = N \int dt d\theta\, \tilde{\eta}(\theta,t)\,(\partial_t \eta + \partial_\theta[(I+\alpha u)\eta])$$

$$+ \int dt\, \tilde{u}\,(\dot{u} + \beta u - \beta[I+\alpha u]\eta(\pi,t))$$

response variable

“Nonlinear Cole-Hopf Transform” $\eta \to \psi$

$$S[u, \tilde{u}, \psi, \tilde{\psi}] = N \int dt d\theta \, \tilde{\psi}(\theta, t) \left(\partial_t \psi + \partial_\theta [(I + \alpha u)\psi]\right)$$

$$+ \int dt \, \tilde{u} \left(\dot{u} + \beta u - \beta [I + \alpha u][\tilde{\psi}(\pi, t)\psi(\pi, t) + \psi(\pi, t)] \right)$$

“Nonlinear Cole-Hopf Transform” $\eta \to \psi$

$$S[u, \tilde{u}, \psi, \tilde{\psi}] = N \int dt d\theta \, \tilde{\psi}(\theta, t) \left(\partial_t \psi + \partial_\theta [(I + \alpha u)\psi]\right)$$

$$+ \int dt \, \tilde{u} \left(\dot{u} + \beta u - \beta [I + \alpha u][\tilde{\psi}(\pi, t)\psi(\pi, t) + \psi(\pi, t)]\right) \qquad - \ln Z_0$$

Initial data ↓ $- \ln Z_0$

"Nonlinear Cole-Hopf Transform" $\eta \to \psi$

$$S[u,\tilde{u},\psi,\tilde{\psi}] = N \int dt d\theta\, \tilde{\psi}(\theta,t)\left(\partial_t \psi + \partial_\theta[(I+\alpha u)\psi]\right)$$

$$+ \int dt\, \tilde{u} \left(\dot{u} + \beta u - \beta[I+\alpha u][\tilde{\psi}(\pi,t)\psi(\pi,t) + \psi(\pi,t)]\right) \qquad - \ln Z_0$$

Initial data ↓ $- \ln Z_0$

$$S = N \left(\frac{1}{2}\tilde{v}\Delta^{-1}v + \text{nonlinear terms}\right)$$

“Nonlinear Cole-Hopf Transform” $\eta \to \psi$

$$S[u,\tilde{u},\psi,\tilde{\psi}] = N\int dt d\theta\, \tilde{\psi}(\theta,t)\,(\partial_t\psi + \partial_\theta[(I+\alpha u)\psi])$$

$$+\int dt\,\tilde{u}\left(\dot{u} + \beta u - \beta[I+\alpha u][\tilde{\psi}(\pi,t)\psi(\pi,t) + \psi(\pi,t)]\right) \qquad - \ln Z_0$$

Initial data ↓ $-\ln Z_0$

$$S = N\left(\frac{1}{2}\tilde{v}\Delta^{-1}v + \text{nonlinear terms}\right)$$

$$\int \mathcal{D}\tilde{v}\mathcal{D}v\,(v^n\tilde{v}^m)e^{-S[v,\tilde{v}]}$$

“Nonlinear Cole-Hopf Transform” $\eta \to \psi$

Initial data

$$S[u,\tilde{u},\psi,\tilde{\psi}] = N \int dt d\theta\, \tilde{\psi}(\theta,t)\left(\partial_t \psi + \partial_\theta[(I+\alpha u)\psi]\right)$$

$$+ \int dt\, \tilde{u}\left(\dot{u} + \beta u - \beta[I+\alpha u][\tilde{\psi}(\pi,t)\psi(\pi,t) + \psi(\pi,t)]\right) \qquad - \ln Z_0$$

$$S = N\left(\frac{1}{2}\tilde{v}\Delta^{-1}v + \text{nonlinear terms}\right)$$

$$\int \mathcal{D}\tilde{v}\mathcal{D}v\,(v^n \tilde{v}^m) e^{-S[v,\tilde{v}]}$$

Laplace’s method in *1/N*

"Nonlinear Cole-Hopf Transform" $\eta \to \psi$

$$S[u,\tilde{u},\psi,\tilde{\psi}] = N\int dt d\theta\, \tilde{\psi}(\theta,t)\,(\partial_t\psi + \partial_\theta[(I+\alpha u)\psi])$$

$$+\int dt\,\tilde{u}\left(\dot{u} + \beta u - \beta[I+\alpha u][\tilde{\psi}(\pi,t)\psi(\pi,t) + \psi(\pi,t)]\right)$$

Initial data

↓

$$-\ln Z_0$$

$$S = N\left(\frac{1}{2}\tilde{v}\Delta^{-1}v + \text{nonlinear terms}\right)$$

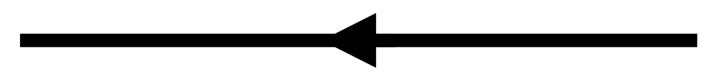
$$\int \mathcal{D}\tilde{v}\mathcal{D}v\,(v^n\tilde{v}^m)e^{-S[v,\tilde{v}]}$$

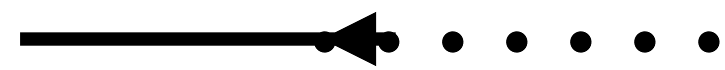
Laplace's method in *1/N*

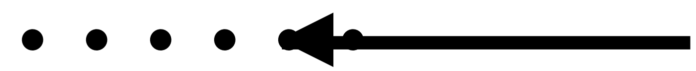
Linear Response

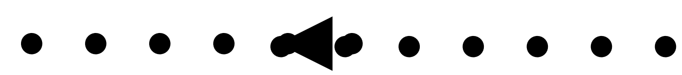
$$
\begin{aligned}
\left(\frac{d}{dt}+\beta\right)\Delta_u^u-\beta\rho(\pi,t)\Delta_u^u-\beta(I+\alpha\bar{u})\Delta_\psi^u &= \delta(t-t') \\
\left(\frac{d}{dt}+\beta\right)\Delta_u^\psi-\beta\rho(\pi,t)\Delta_u^\psi-\beta(I+\alpha\bar{u})\Delta_\psi^\psi &= 0 \\
\partial_t\Delta_\psi^u+\partial_\theta\left[(I+\alpha\bar{u})\Delta_\psi^u\right]+\partial_\theta\rho\Delta_u^u &= 0 \\
\partial_t\Delta_\psi^\psi+\partial_\theta\left[(I+\alpha\bar{u})\Delta_\psi^\psi\right]+\partial_\theta\rho\Delta_u^\psi &= \frac{1}{N}\delta(\theta-\theta')\delta(t-t')
\end{aligned}
$$

Linear Response

$$\Delta_u^u =$$ 

$$\Delta_u^\psi =$$ 

$$\Delta_\psi^u =$$ 

$$\Delta_\psi^\psi =$$ 

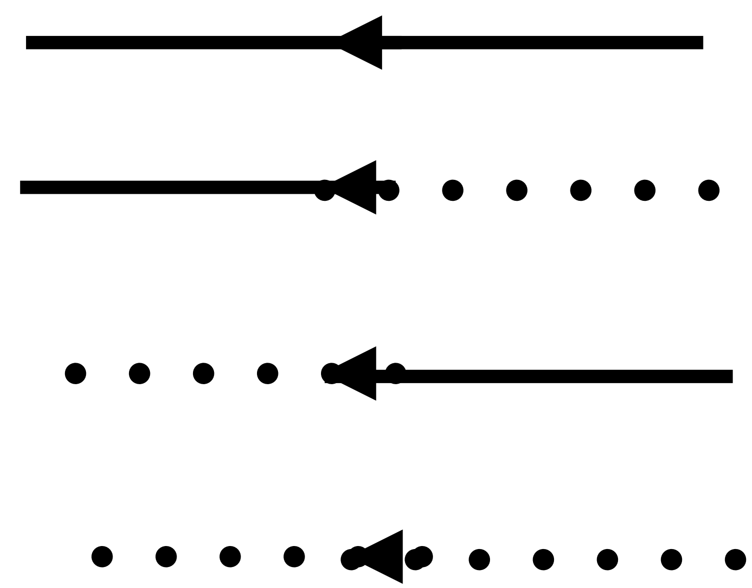
Linear Response

$$\Delta_u^u \quad =$$

$$\Delta_u^\psi \quad =$$

$$\Delta_\psi^u \quad =$$

$$\Delta_\psi^\psi \quad =$$

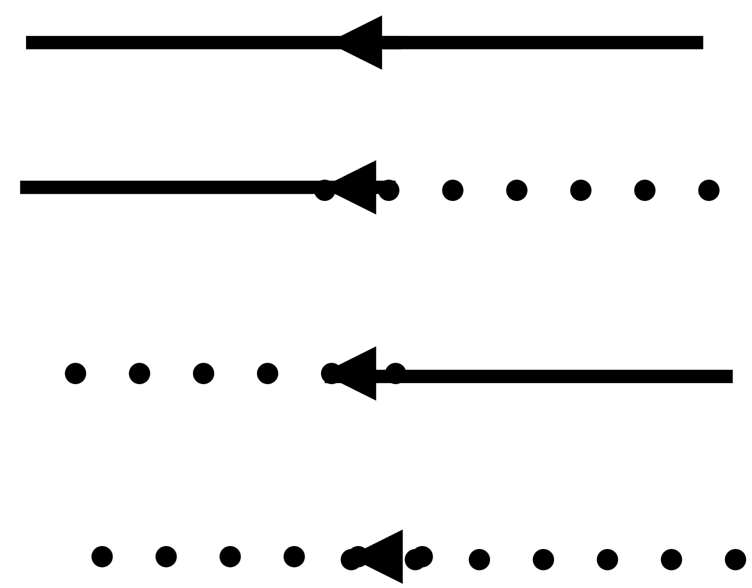


$$\langle \delta u(t) \delta u(t') \rangle$$

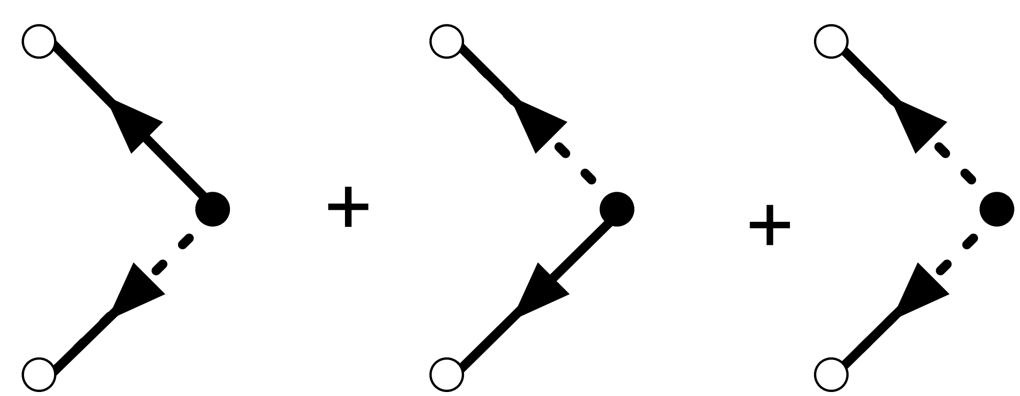
$$\delta u = u - \bar{u}$$

Linear Response

$$\Delta_u^u =$$
$$\Delta_u^\psi =$$
$$\Delta_\psi^u =$$
$$\Delta_\psi^\psi =$$



$$\langle \delta u(t) \delta u(t') \rangle =$$



$$\delta u = u - \bar{u}$$

Steady state

$$\dot{u} = -\beta u + \beta(I + \alpha u)\rho(\pi, t) \quad = 0$$

$$\partial_t \rho = -\partial_\theta \left[(I(t) + \alpha u(t))\rho\right] \qquad = 0$$

$$\bar{\rho} = \frac{1}{2\pi} \qquad \qquad \bar{u} = \frac{I}{2\pi}\left(1 - \frac{\alpha}{2\pi}\right)^{-1}$$

$$\nu = (I + \alpha\bar{u})\bar{\rho} = \bar{u}$$

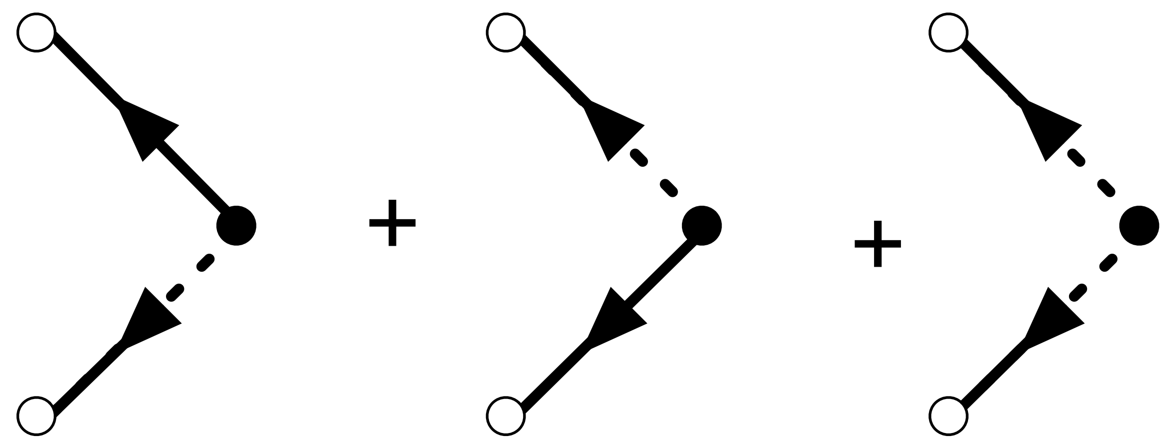
Drive Correlations

$\langle \delta u(t) \delta u(t') \rangle$ $\delta u = u - \bar{u}$

Drive Correlations

$\langle \delta u(t) \delta u(t') \rangle \qquad \delta u = u - \bar{u}$

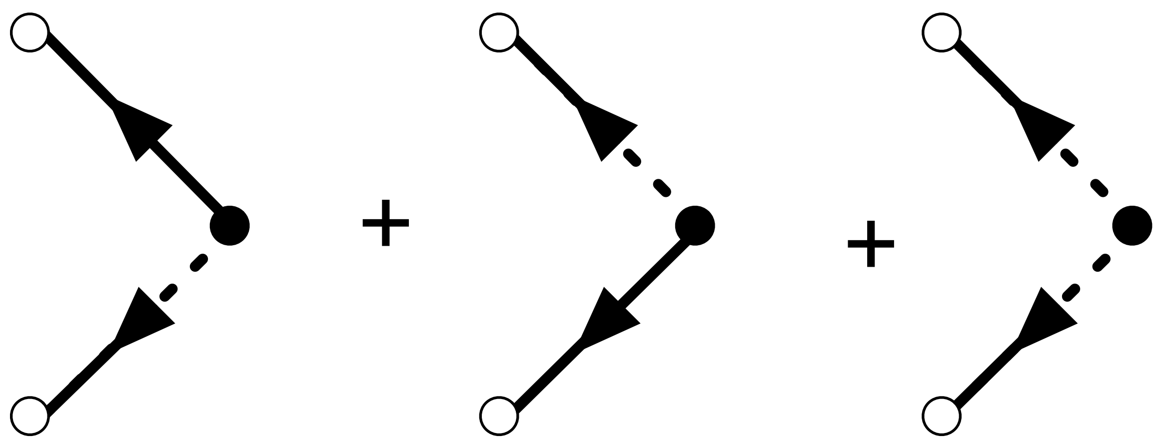
$=$



Drive Correlations

$$\langle \delta u(t)\delta u(t')\rangle \qquad \delta u = u - \bar{u}$$

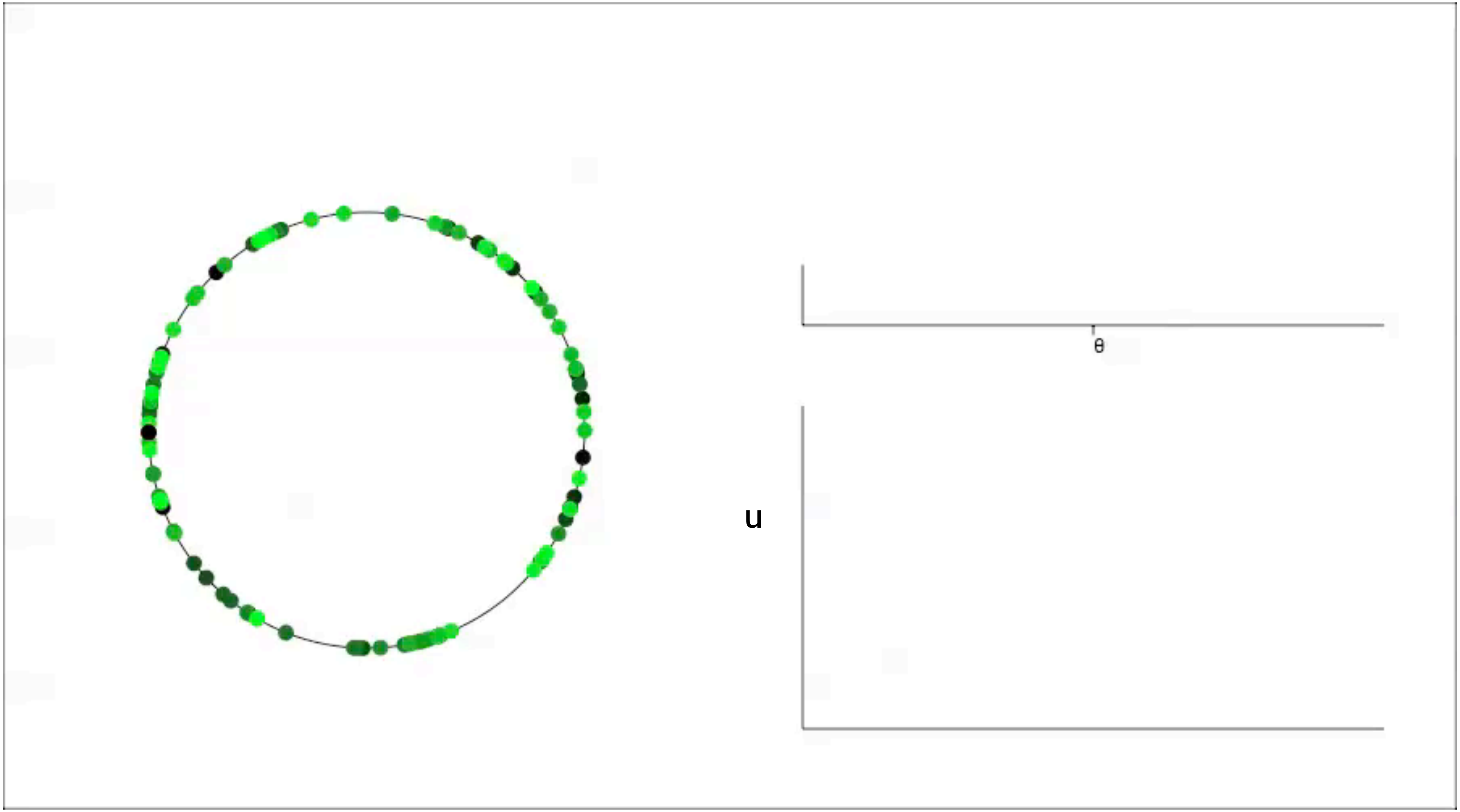
$=$



$$= \quad \beta \int dt'' \;\; (I + \alpha\bar{u}(t''))\, \Delta^u_u(t,t'')\Delta^\psi_u(t',\pi,t'')\rho(\pi,\alpha,t'') + (t \leftrightarrow t')$$

$$- \quad \frac{N}{(2\pi)^2}\int d\theta\, \Delta^\psi_u(t,s)\int d\theta'\, \Delta^\psi_u(t',s') \quad + O\left(\frac{1}{N^2}\right)$$

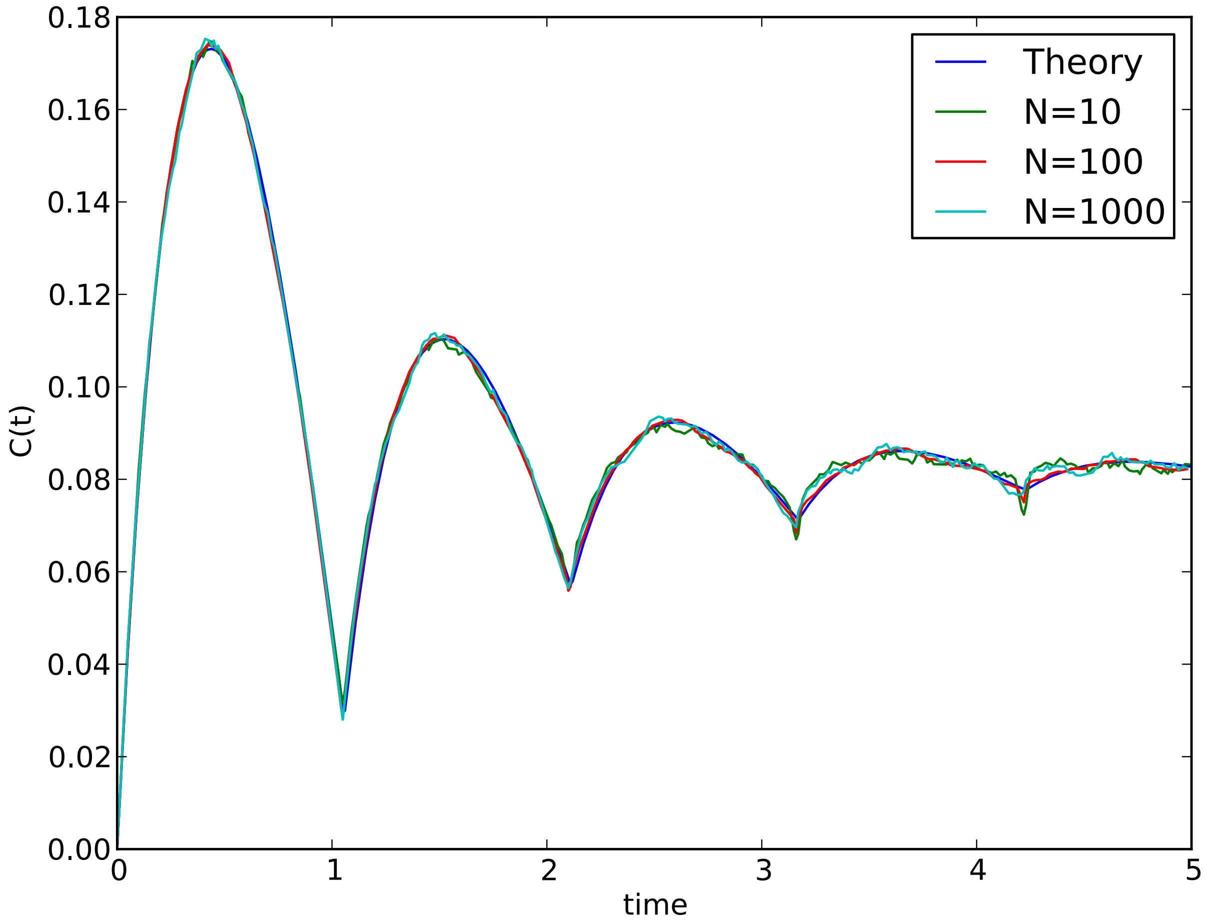
Drive Correlations

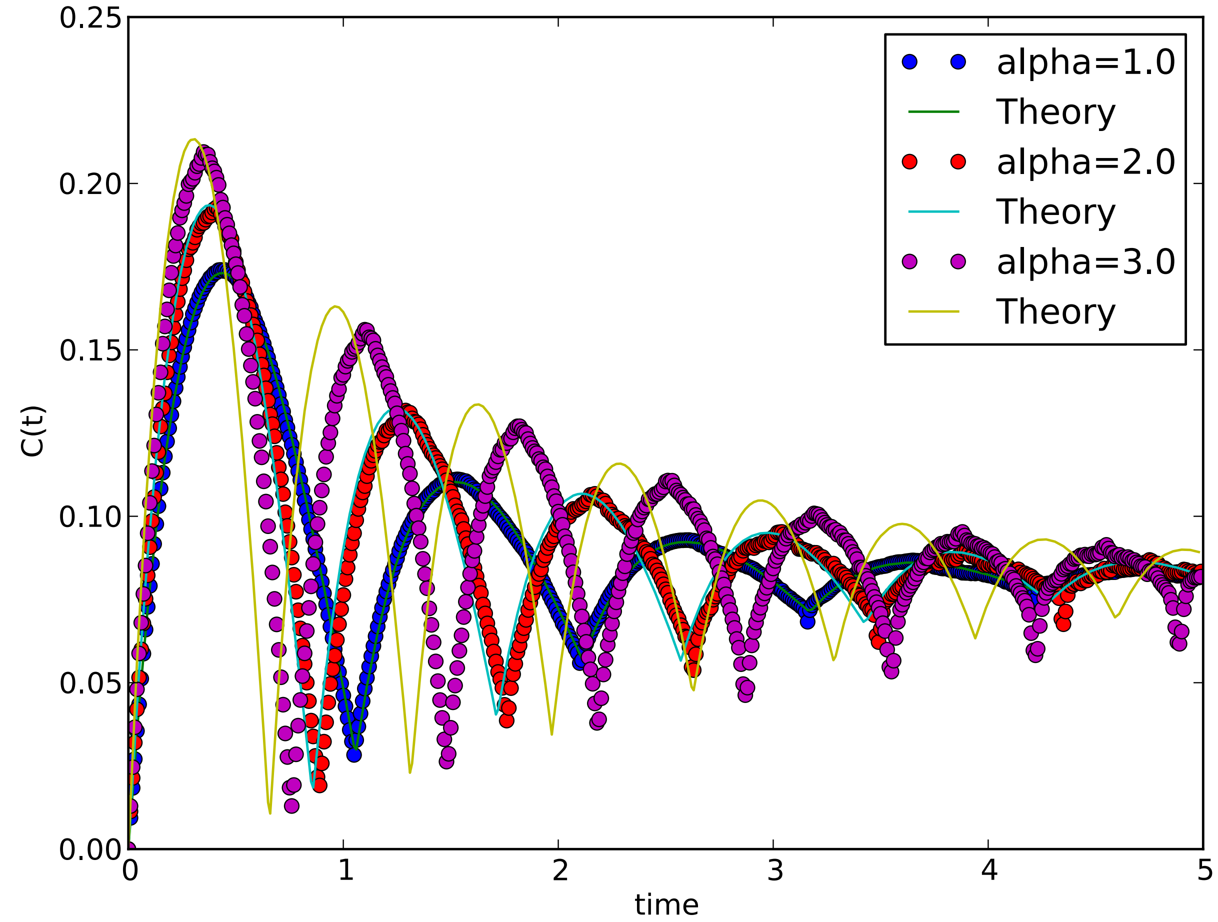


Drive Correlations

$$\begin{aligned}\langle \delta u(t)^2 \rangle \quad = \quad & \frac{1}{N} \sum_{k=0}^{\infty} \left(1 - \frac{1}{2}\delta_{k,0}\right) \frac{\beta^2}{\pi\delta} \left(I + \alpha \bar{u}_0\right) \\ & \times e^{-\beta\delta\Delta t_k} \left[1 - e^{-2\beta\delta(t - t_0 - \Delta t_k)}\right] H(t - t_0 - \Delta t_k) \\ & - \frac{1}{N} \bar{u}_0^2 \left(1 - e^{-\beta\delta(t-t_0)}\right)^2\end{aligned}$$

Correlation transients

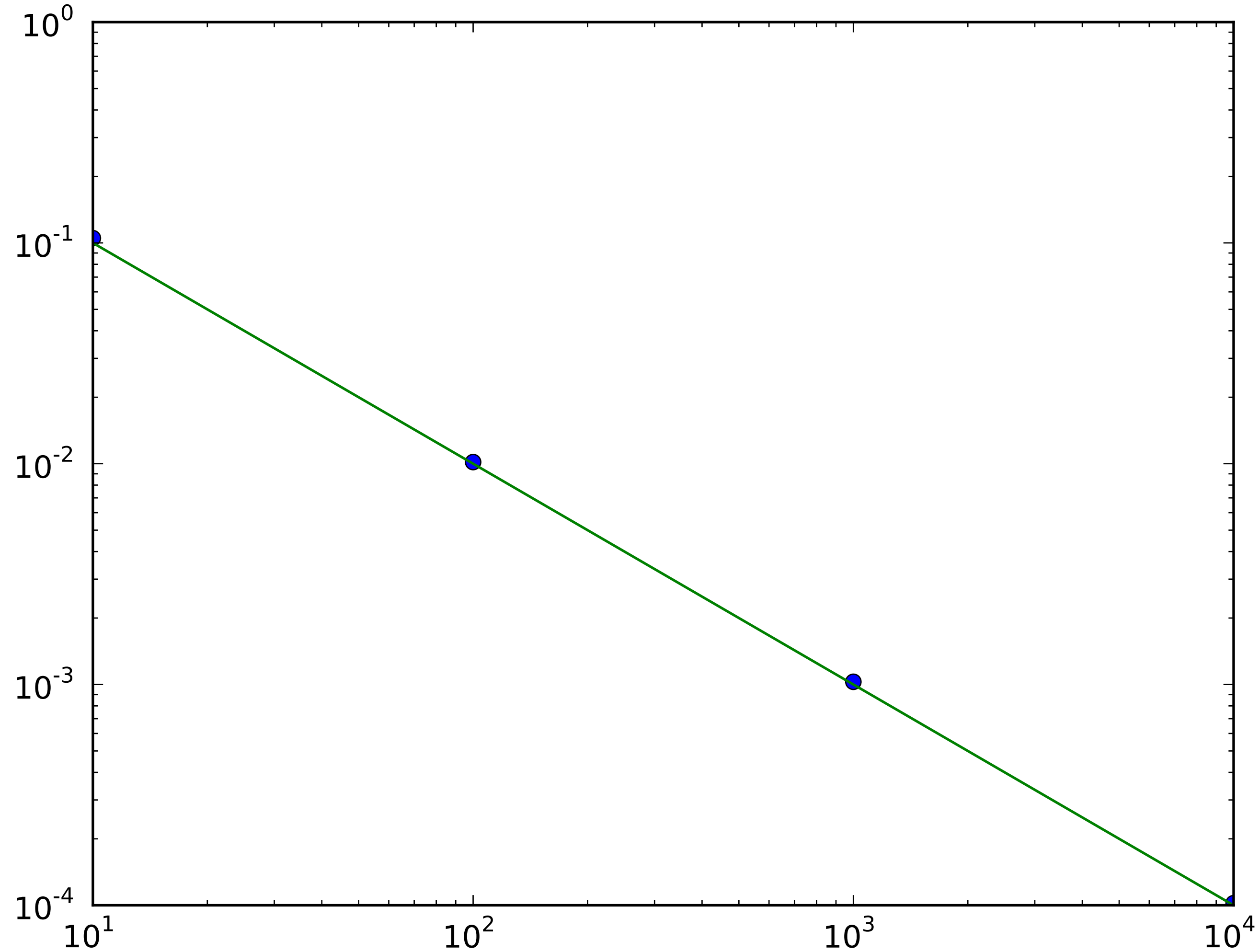




$$C(t) = \langle u(t)^2 \rangle - \langle u(t) \rangle^2 \propto \frac{1}{N}$$

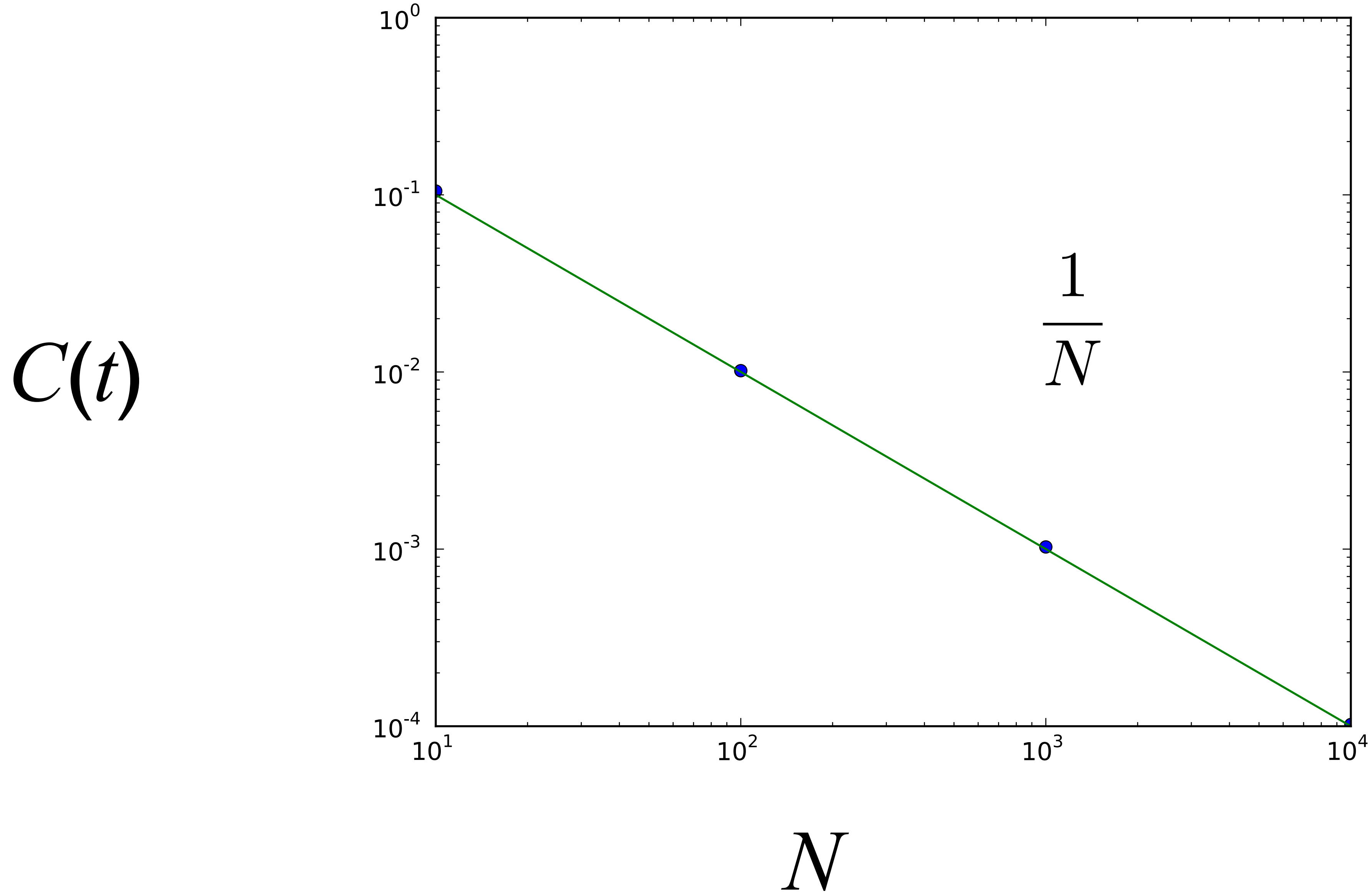
Correlation asymptotic state

$C(t)$



N

Correlation asymptotic state



Firing rate fluctuations

$$\nu(t) = (I(t) + \alpha u(t))\eta(\pi, t)$$

$$\langle \nu(t) \rangle = (I(t) + \alpha u(t))\bar{\rho} = \bar{u}$$

$$\langle (\nu(t) - \bar{u}) \left(\nu(t') - \bar{u}\right) \rangle$$

Firing rate fluctuations

$$\nu(t) = (I(t) + \alpha u(t))\eta(\pi, t)$$

$$\langle \nu(t) \rangle = (I(t) + \alpha u(t))\bar{\rho} = \bar{u}$$

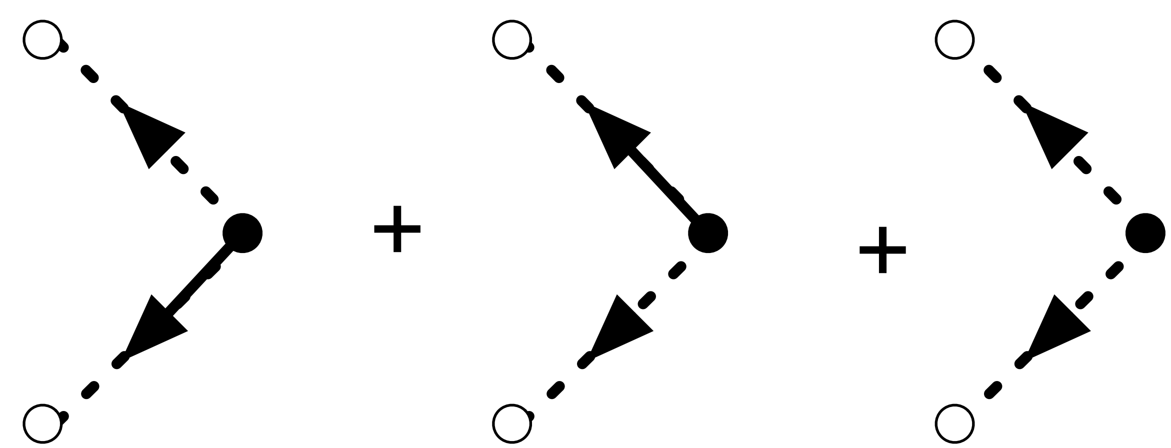
$$\langle (\nu(t) - \bar{u}) (\nu(t') - \bar{u}) \rangle \quad =$$

Firing rate fluctuations

$$\nu(t) = (I(t) + \alpha u(t))\eta(\pi, t)$$

$$\langle \nu(t) \rangle = (I(t) + \alpha u(t))\bar{\rho} = \bar{u}$$

$$\langle (\nu(t) - \bar{u})\,(\nu(t') - \bar{u}) \rangle \quad = $$

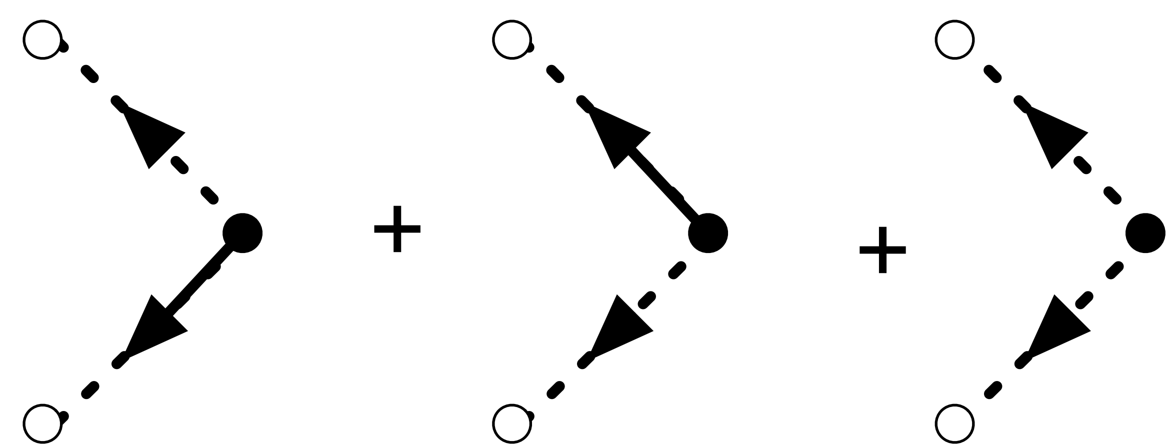


Firing rate fluctuations

$$\nu(t) = (I(t) + \alpha u(t))\eta(\pi, t)$$

$$\langle \nu(t) \rangle = (I(t) + \alpha u(t))\bar{\rho} = \bar{u}$$

$$\langle (\nu(t) - \bar{u}) (\nu(t') - \bar{u}) \rangle \quad =$$



$$= (I + \alpha \bar{u})^2 \langle \eta(\pi, t) \eta(\pi, t') \rangle$$

Firing rate fluctuations

$$\nu(t) = (I(t) + \alpha u(t))\eta(\pi, t)$$

$$\langle \nu(t) \rangle = (I(t) + \alpha u(t))\bar{\rho} = \bar{u}$$

$$\langle (\nu(t) - \bar{u})\,(\nu(t') - \bar{u}) \rangle \quad = (I + \alpha \bar{u})^2 \, \langle \eta(\pi, t)\eta(\pi, t') \rangle$$

Firing rate fluctuations

$$\nu(t) = (I(t) + \alpha u(t))\eta(\pi, t)$$

$$\langle \nu(t) \rangle = (I(t) + \alpha u(t))\bar{\rho} = \bar{u}$$

$$\langle (\nu(t) - \bar{u})(\nu(t') - \bar{u}) \rangle \quad = (I + \alpha \bar{u})^2 \langle \eta(\pi, t)\eta(\pi, t') \rangle$$

$$=$$

Firing rate fluctuations

$$\nu(t) = (I(t) + \alpha u(t))\eta(\pi, t)$$

$$\langle \nu(t) \rangle = (I(t) + \alpha u(t))\bar{\rho} = \bar{u}$$

$$\langle (\nu(t) - \bar{u})(\nu(t') - \bar{u}) \rangle \quad = (I + \alpha \bar{u})^2 \langle \eta(\pi, t)\eta(\pi, t') \rangle$$

$$= \quad \frac{\bar{u}}{N dt}$$

Firing rate fluctuations

$$\nu(t) = (I(t) + \alpha u(t))\eta(\pi, t)$$

$$\langle \nu(t) \rangle = (I(t) + \alpha u(t))\bar{\rho} = \bar{u}$$

$$\langle (\nu(t) - \bar{u})(\nu(t') - \bar{u}) \rangle \quad = (I + \alpha \bar{u})^2 \langle \eta(\pi, t)\eta(\pi, t') \rangle$$

$$= \frac{\bar{u}}{N dt}$$

Poisson behavior

Firing rate fluctuations

$$\nu(t) = (I(t) + \alpha u(t))\eta(\pi, t)$$

$$\langle \nu(t) \rangle = (I(t) + \alpha u(t))\bar{\rho} = \bar{u}$$

$$\langle (\nu(t) - \bar{u}) (\nu(t') - \bar{u}) \rangle \quad = (I + \alpha \bar{u})^2 \langle \eta(\pi, t)\eta(\pi, t') \rangle$$

$$= \qquad \frac{\bar{u}}{N dt} \qquad -$$

Poisson behavior

Firing rate fluctuations

$$\nu(t) = (I(t) + \alpha u(t))\eta(\pi, t)$$

$$\langle \nu(t) \rangle = (I(t) + \alpha u(t))\bar{\rho} = \bar{u}$$

$$\langle (\nu(t) - \bar{u})(\nu(t') - \bar{u}) \rangle \quad = (I + \alpha\bar{u})^2 \langle \eta(\pi, t)\eta(\pi, t') \rangle$$

$$= \quad \frac{\bar{u}}{N dt} \quad - \quad \frac{\bar{u}^2}{N}$$

Poisson behavior

Firing rate fluctuations

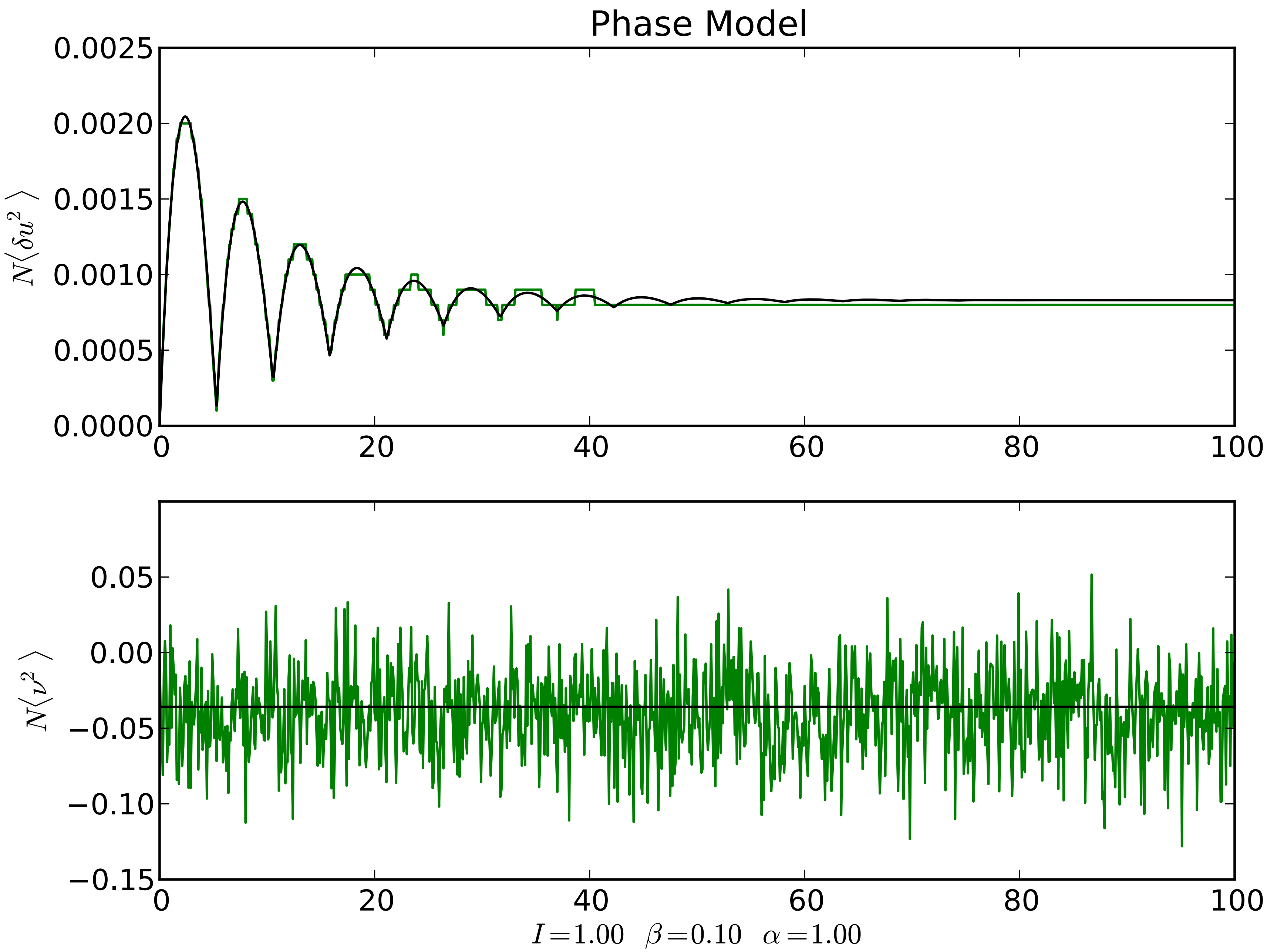
$$\nu(t) = (I(t) + \alpha u(t))\eta(\pi, t)$$

$$\langle \nu(t) \rangle = (I(t) + \alpha u(t))\bar{\rho} = \bar{u}$$

$$\langle (\nu(t) - \bar{u})(\nu(t') - \bar{u}) \rangle \quad = (I + \alpha \bar{u})^2 \langle \eta(\pi, t)\eta(\pi, t') \rangle$$

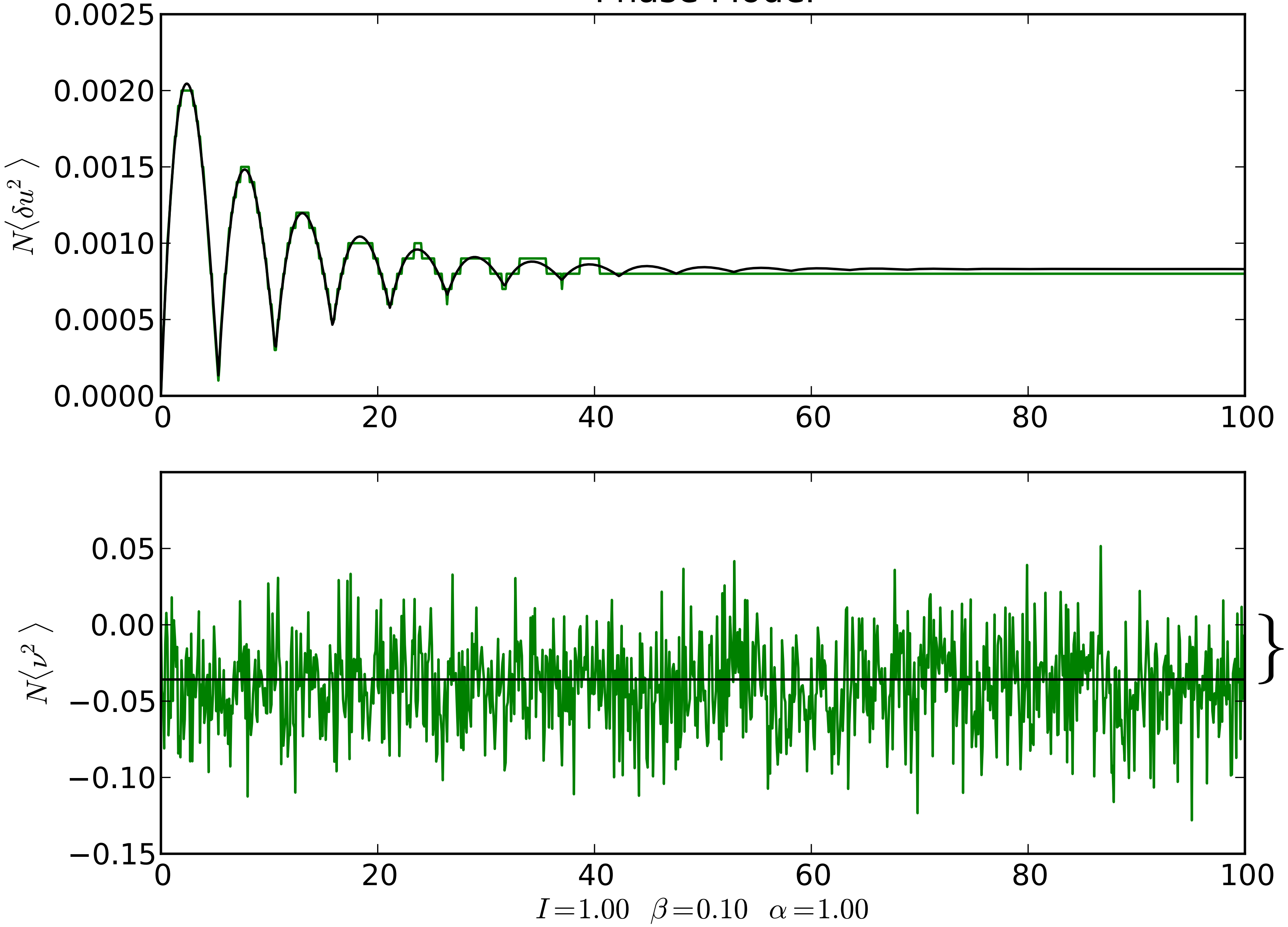
$$= \quad \frac{\bar{u}}{N dt} \quad - \quad \frac{\bar{u}^2}{N}$$

Poisson behavior sampling noise

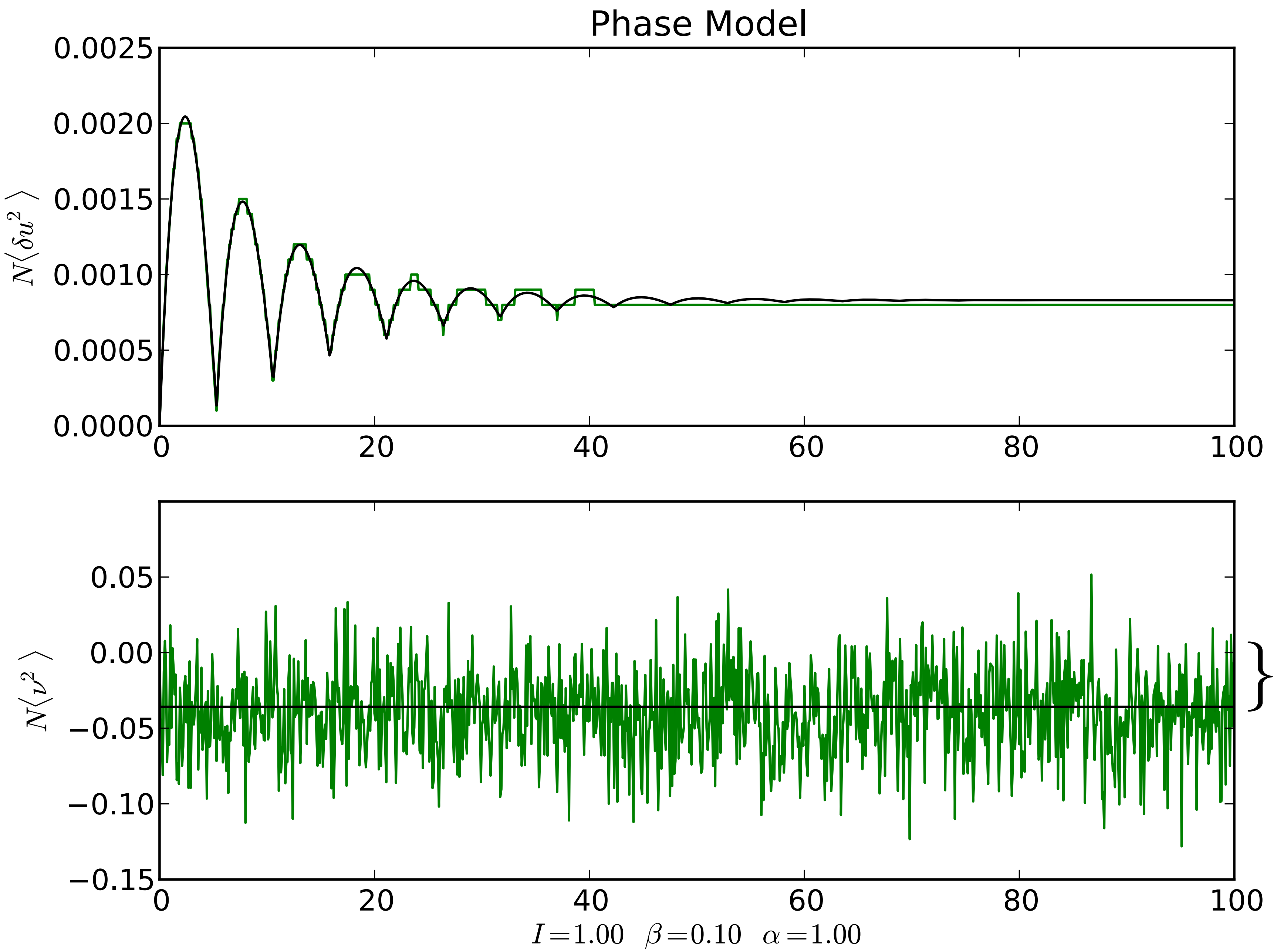


Phase Model
0.0025
0.0020
0.0015
0.0010
0.0005
0.0000
$N\langle \delta u^2 \rangle$
0
20
40
60
80
100
0.05
0.00
−0.05
−0.10
−0.15
$N\langle \nu^2 \rangle$
$I = 1.00 \quad \beta = 0.10 \quad \alpha = 1.00$

Phase Model



$N\langle \delta u^2 \rangle$
0.0025
0.0020
0.0015
0.0010
0.0005
0.0000
0
20
40
60
80
100
$N\langle \nu^2 \rangle$
0.05
0.00
−0.05
−0.10
−0.15
0
20
40
60
80
100
$I = 1.00 \quad \beta = 0.10 \quad \alpha = 1.00$



Deviation from Poisson

Theta Model

$$\dot{\theta}_i(t) = 1 - \cos\theta_i(t) + (I_i(t) + \alpha_i u(t))(1 + \cos\theta_i(t))$$

$$\dot{u}_i + \beta u_i = \frac{\beta}{N}\sum_j \delta(t - t_j^s)$$

Theta Model

$$\dot{\theta}_i(t) = 1 - \cos\theta_i(t) + (I_i(t) + \alpha_i u(t))(1 + \cos\theta_i(t))$$

$$\dot{u}_i + \beta u_i = \frac{\beta}{N} \sum_j \delta(t - t_j^s)$$

$$S = S[\tilde{u}(t), u(t)] + S[\tilde{\varphi}(\theta, t), \varphi(\theta, t)]$$

Theta Model

$$\dot{\theta}_i(t) = 1 - \cos\theta_i(t) + (I_i(t) + \alpha_i u(t))(1 + \cos\theta_i(t))$$

$$\dot{u}_i + \beta u_i = \frac{\beta}{N}\sum_j \delta(t - t_j^s)$$

$$S = S[\tilde{u}(t), u(t)] + S[\tilde{\varphi}(\theta, t), \varphi(\theta, t)]$$

$$\begin{aligned} S[\varphi, \tilde{\varphi}] \quad = \quad & N \int dt d\theta\, \tilde{\varphi}(\theta, t) \left[\partial_t \varphi(\theta, t) + \partial_\theta \left[1 - \cos\theta \right.\right. \\ & \left.\left. + (1 + \cos\theta) \left\{I + \alpha u(t)\right\} \varphi(\theta, t)\right]\right] - \ln Z[\tilde{\varphi}_0(\theta, t_0)] \end{aligned}$$

Theta Model

$$\dot{\theta}_i(t) = 1 - \cos\theta_i(t) + (I_i(t) + \alpha_i u(t))(1 + \cos\theta_i(t))$$

$$\dot{u}_i + \beta u_i = \frac{\beta}{N}\sum_j \delta(t - t_j^s)$$

$$S = S[\tilde{u}(t), u(t)] + S[\tilde{\varphi}(\theta, t), \varphi(\theta, t)]$$

$$\begin{aligned} S[\varphi, \tilde{\varphi}] &= N\int dt d\theta\, \tilde{\varphi}(\theta, t)\left[\partial_t\varphi(\theta, t) + \partial_\theta\left[1 - \cos\theta\right.\right. \\ &\quad \left.\left. +(1 + \cos\theta)\left\{I + \alpha u(t)\right\}\varphi(\theta, t)\right]\right] - \ln Z[\tilde{\varphi}_0(\theta, t_0)] \end{aligned}$$

$$\begin{aligned} S[\tilde{u}(t), u(t)] &= \int_{t_0}^{t} ds\, \tilde{u}(s)\left(\frac{d}{ds}u(s) + \beta u(s)\right. \\ &\quad \left. -2\beta\left\{\tilde{\varphi}(\pi, s)\varphi(\pi, s) + \varphi(\pi, s)\right\}\right) - \ln Z[\tilde{u}(t_0)] \end{aligned}$$

Steady state

$$\rho_0(\theta) = \frac{\sqrt{I + u_0}}{\pi(1 - \cos\theta + (I + \alpha u_0)(1 + \cos\theta))}$$

$$u_0 = \sqrt{I + \alpha u_0}$$

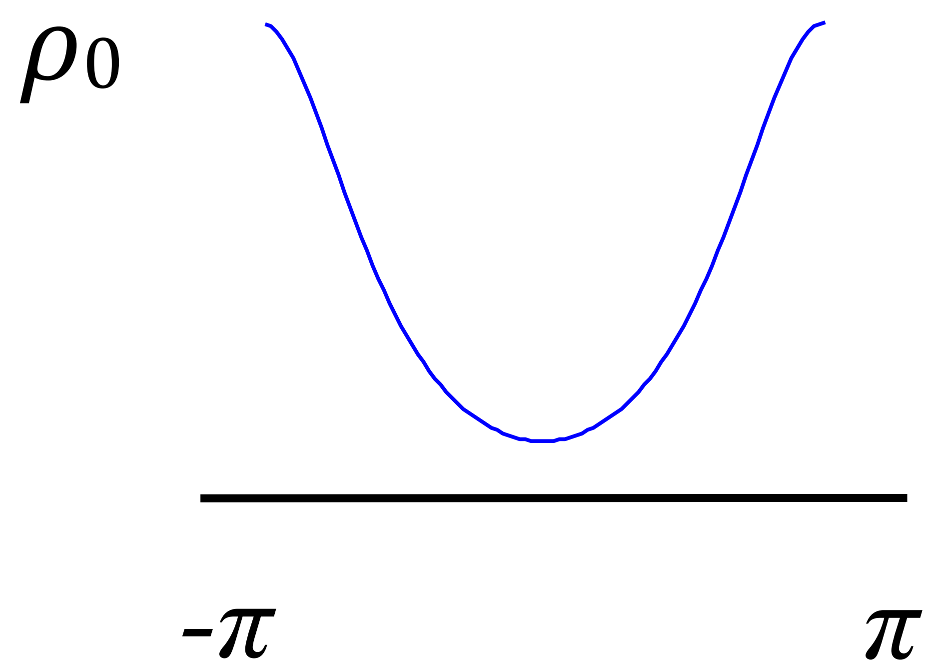
$$\nu = \frac{1}{\pi}\sqrt{I + \alpha u_0}$$

Steady state

$$\rho_0(\theta) = \frac{\sqrt{I + u_0}}{\pi(1 - \cos\theta + (I + \alpha u_0)(1 + \cos\theta))}$$

$$u_0 = \sqrt{I + \alpha u_0}$$

$$\nu = \frac{1}{\pi}\sqrt{I + \alpha u_0}$$



Firing rate fluctuations

$$\langle \nu(t) \rangle = 2\rho(\pi, t)$$

Firing rate fluctuations

$$\langle \nu(t) \rangle = 2\rho(\pi, t)$$

$$\langle \nu(t) \rangle = \int d\alpha d\Omega d\alpha' d\Omega' \langle \psi(x_\pi)\psi(x'_\pi) \rangle + \frac{1}{Ndt} \langle \nu(t) \rangle$$

Firing rate fluctuations

$$\langle \nu(t) \rangle = 2\rho(\pi, t)$$

Poisson ↓

$$\langle \nu(t) \rangle = \int d\alpha d\Omega d\alpha' d\Omega' \langle \psi(x_\pi)\psi(x'_\pi) \rangle + \frac{1}{Ndt} \langle \nu(t) \rangle$$

Firing rate fluctuations

$$\langle \nu(t) \rangle = 2\rho(\pi, t)$$

Poissson

$$\langle \nu(t) \rangle = \boxed{\int d\alpha d\Omega d\alpha' d\Omega' \langle \psi(x_\pi) \psi(x'_\pi) \rangle} + \frac{1}{N dt} \langle \nu(t) \rangle$$

Anomalous finite size effects

Firing rate fluctuations

$$\langle \nu(t) \rangle = 2\rho(\pi, t)$$

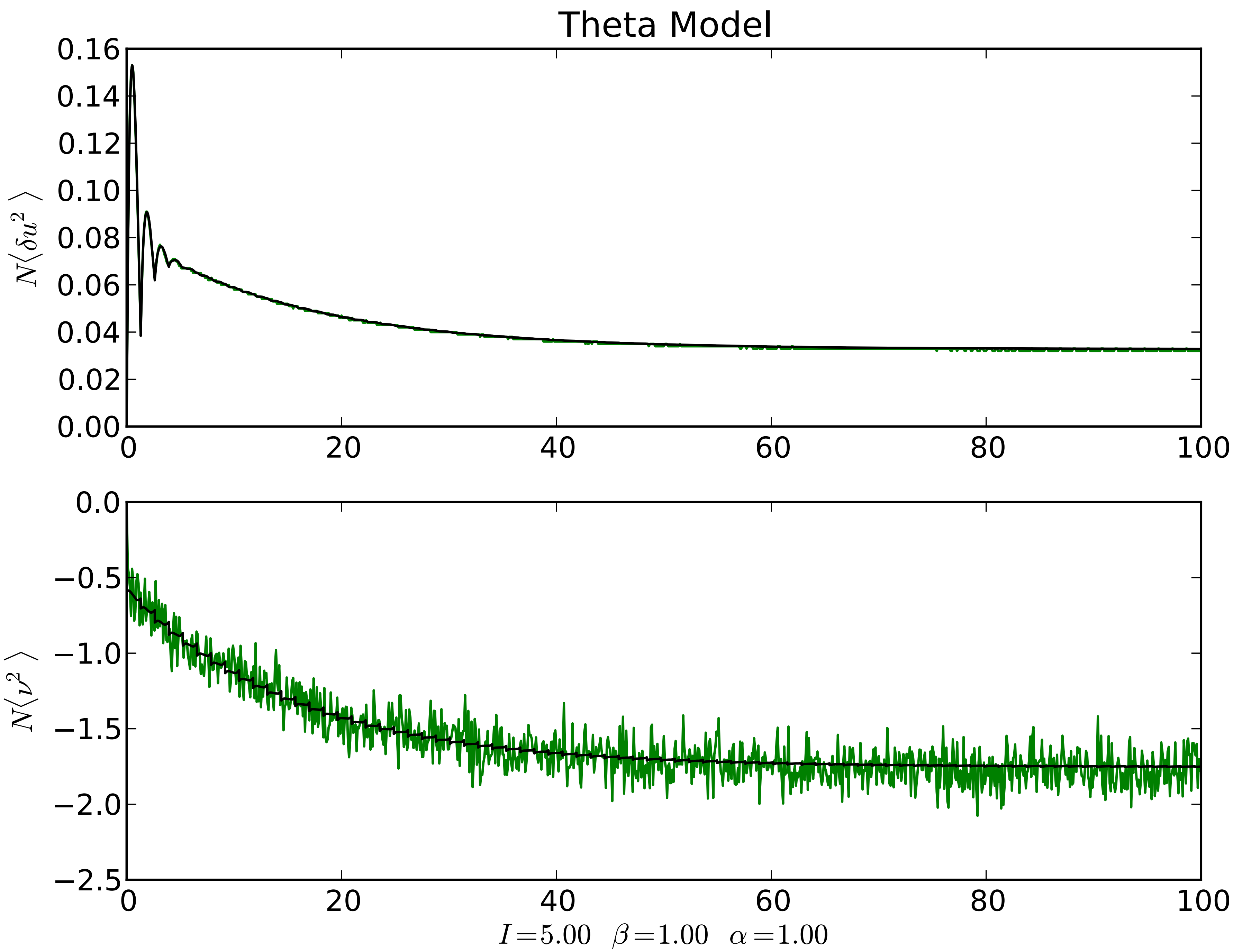
Poisson

$$\langle \nu(t) \rangle = \boxed{\int d\alpha d\Omega d\alpha' d\Omega' \langle \psi(x_\pi)\psi(x'_\pi) \rangle} + \frac{1}{N dt} \langle \nu(t) \rangle$$

Anomalous finite size effects

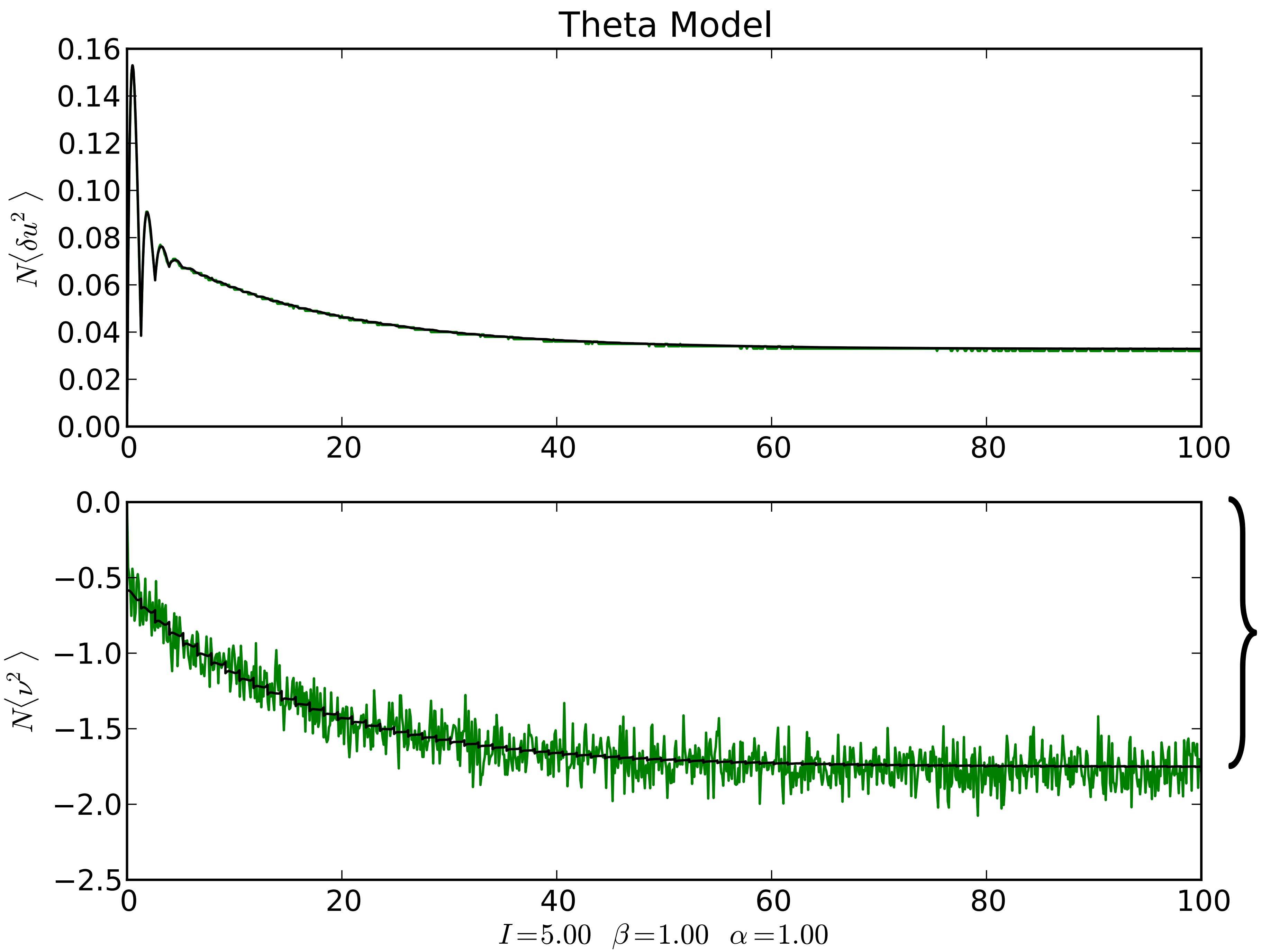
not in phase model

Simulations



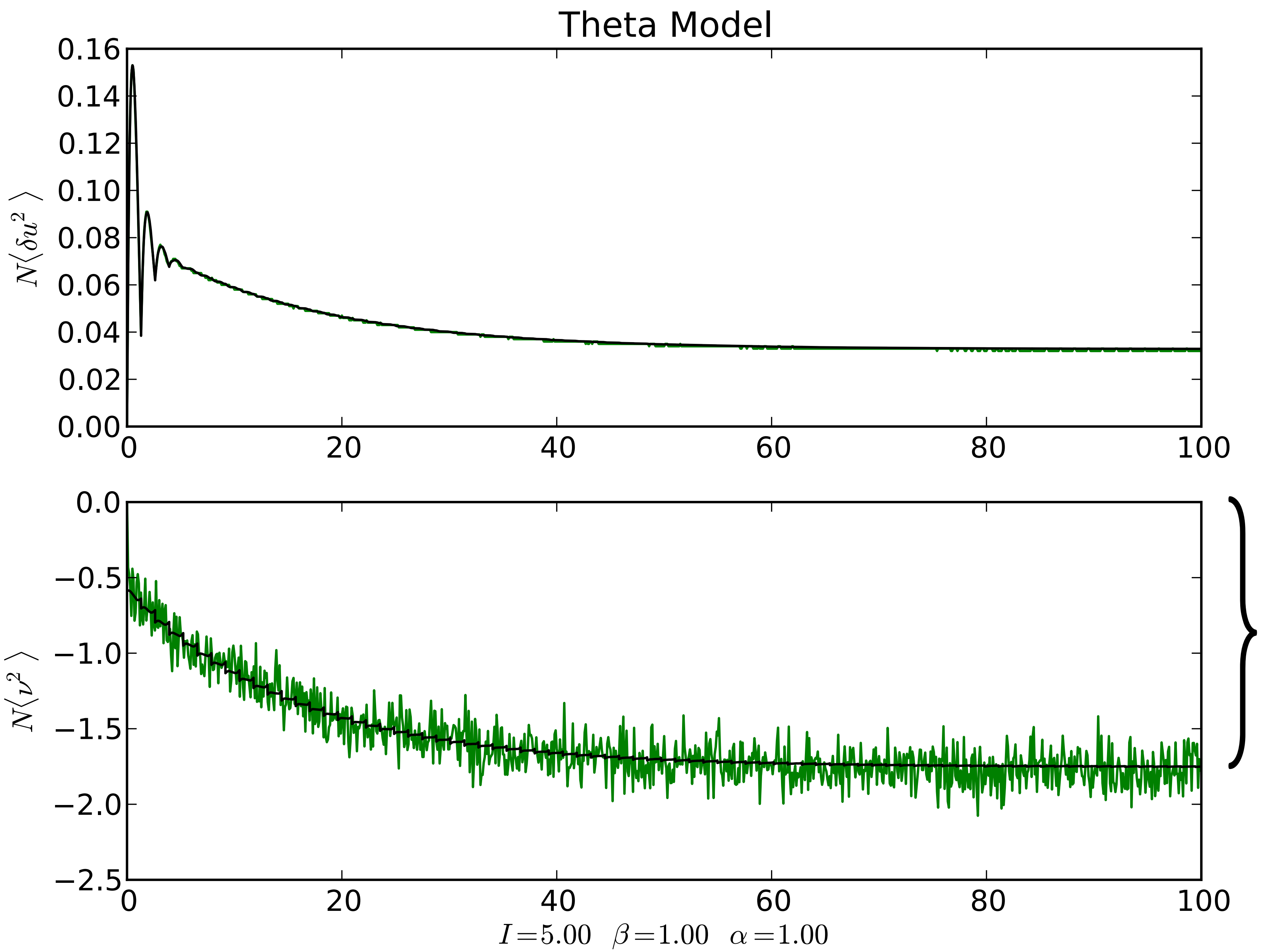
N=1000

Simulations



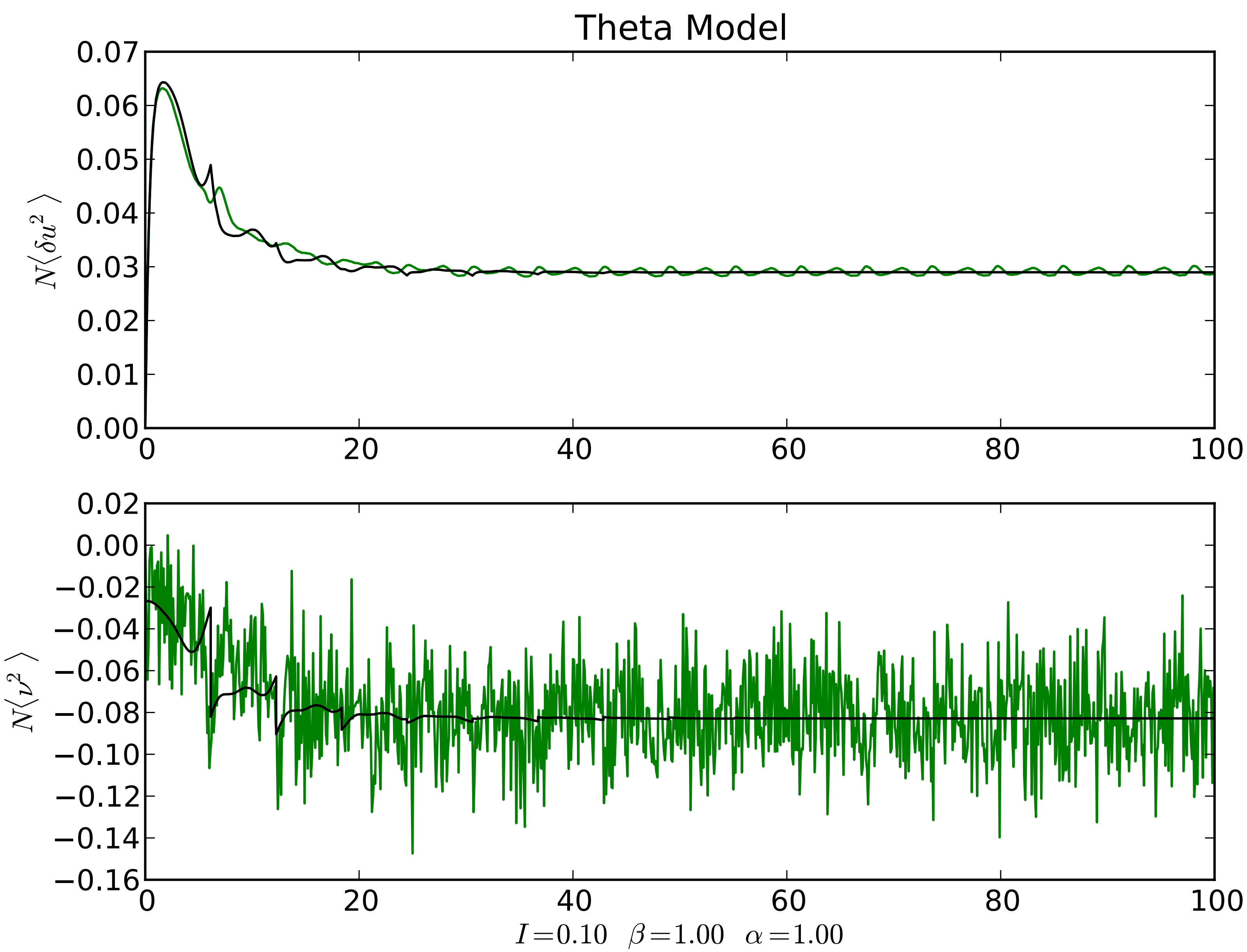
N=1000

Simulations

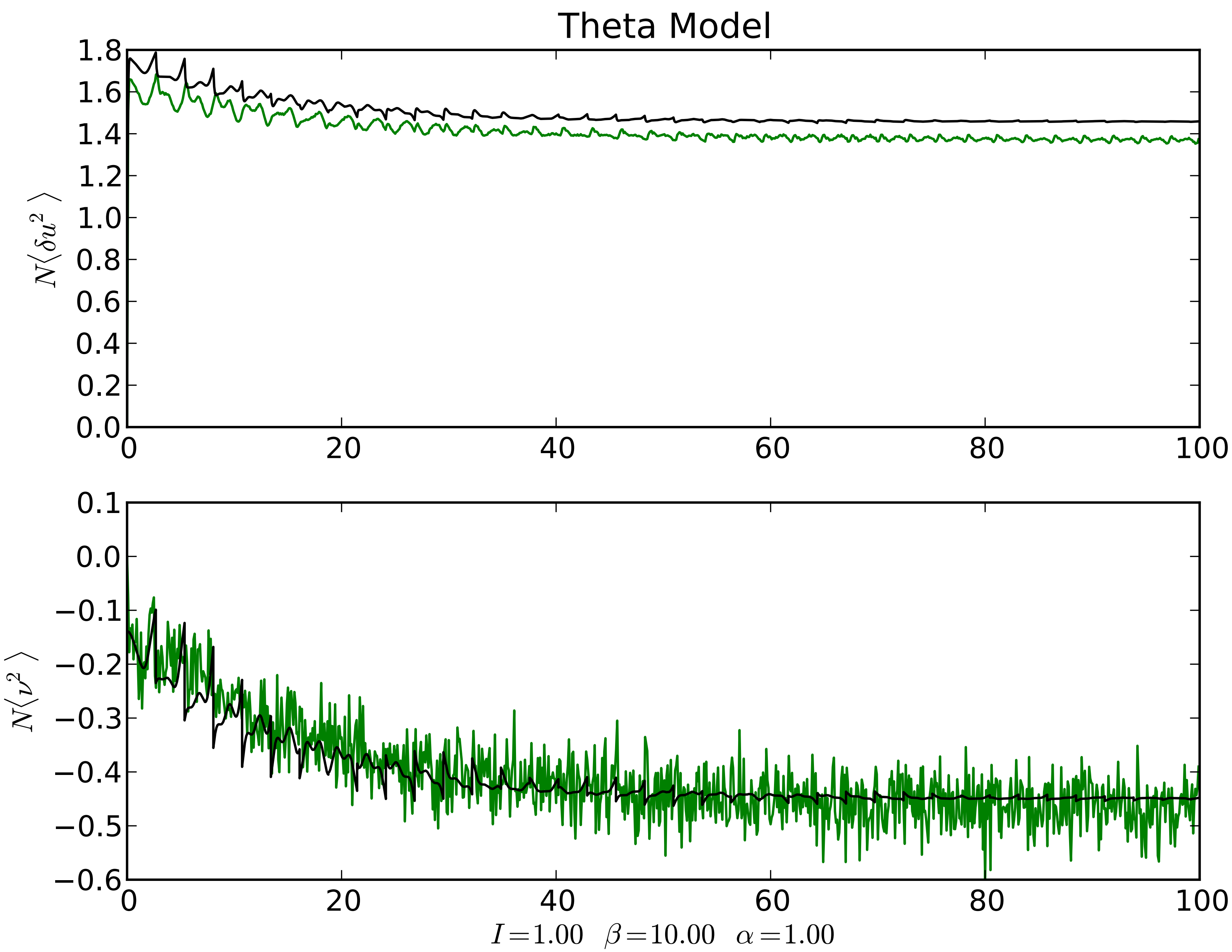


N=1000

deviation from Poisson


Theta Model
$N\langle \delta u^2 \rangle$
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.00
0
20
40
60
80
100
$N\langle \nu^2 \rangle$
0.02
0.00
−0.02
−0.04
−0.06
−0.08
−0.10
−0.12
−0.14
−0.16
0
20
40
60
80
100
$I=0.10 \quad \beta=1.00 \quad \alpha=1.00$

N = 10


Theta Model
$N\langle\delta u^2\rangle$
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
20
40
60
80
100
$N\langle\nu^2\rangle$
0.1
0.0
−0.1
−0.2
−0.3
−0.4
−0.5
−0.6
0
20
40
60
80
100
$I=1.00 \quad \beta=10.00 \quad \alpha=1.00$

N=10

Slides on sciencehouse.wordpress.com