Kinetic Theory of Coupled Oscillators

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Coupled Oscillators

- Coupled oscillators describe many phenomena including dynamics of neural circuits, synchronization of fireflies, crickets, Josephson junctions, ...
- Most analyses focused on small systems or infinitely large (mean field limit)
- Networks that are large but not infinite is not well understood

Kuramoto Model

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N} f(\theta_j - \theta_i) \qquad f(\theta) = \sin(\theta)$$

where ω_i is drawn from a distribution $g(\omega)$

Weak coupling limit (phase of limit cycle)

Interested in dynamics of N large but not infinite

Two oscillator example

$$\frac{d\theta_1}{dt} = \omega_1 + \frac{K}{2}\sin(\theta_2 - \theta_1)$$

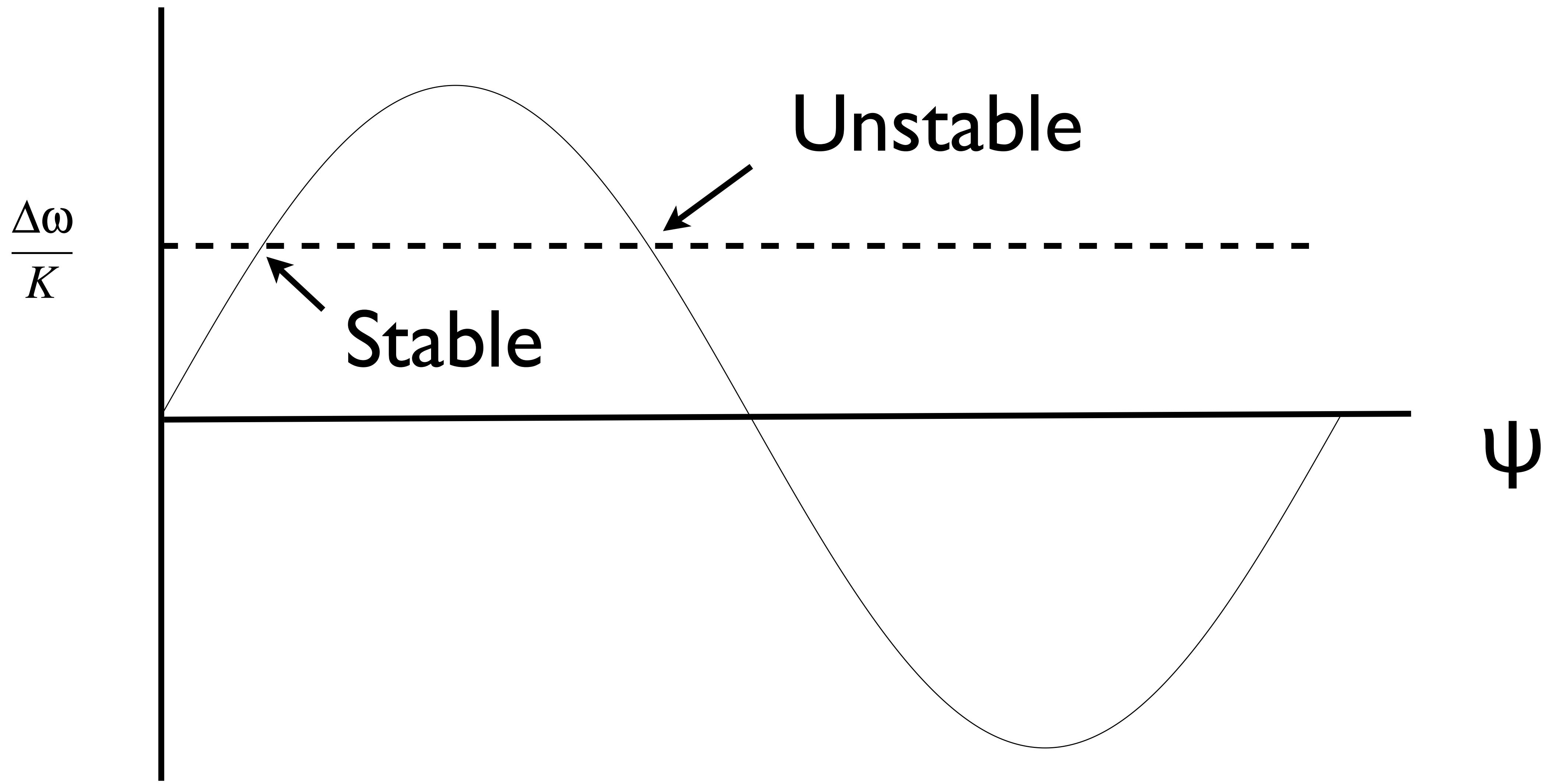
$$\frac{d\theta_2}{dt} = \omega_2 + \frac{K}{2}\sin(\theta_1 - \theta_2)$$

$$\frac{d\psi}{dt} = \Delta\omega - K\sin(\psi)$$

$$\psi = \theta_1 - \theta_2 \qquad \Delta\omega = \omega_1 - \omega_2$$

Fixed points obey $$\frac{\Delta\omega}{K} = \sin(\psi)$$

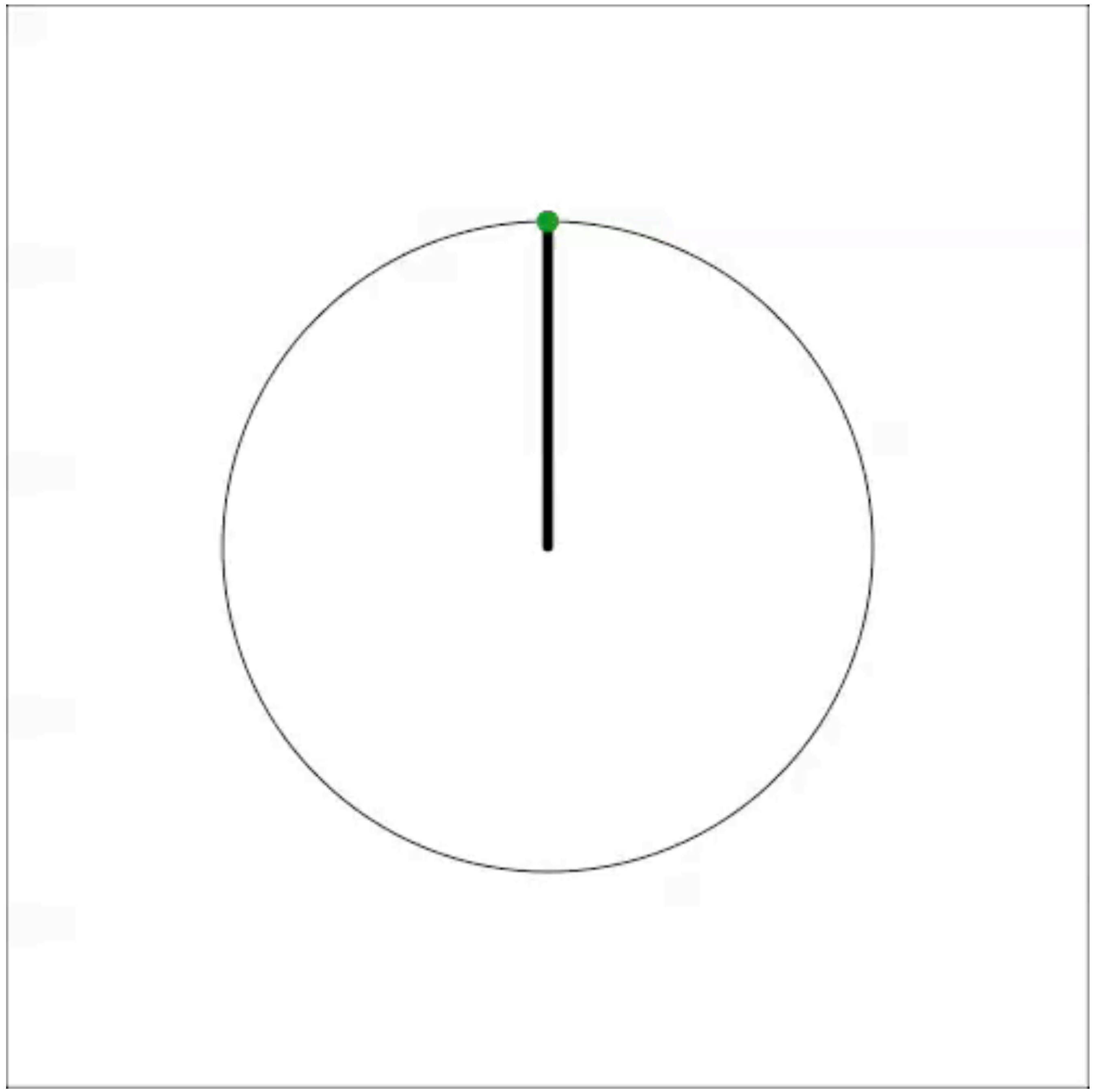
Fixed points at intersection



$K_c = \Delta\omega$ is the critical (bifurcation) point

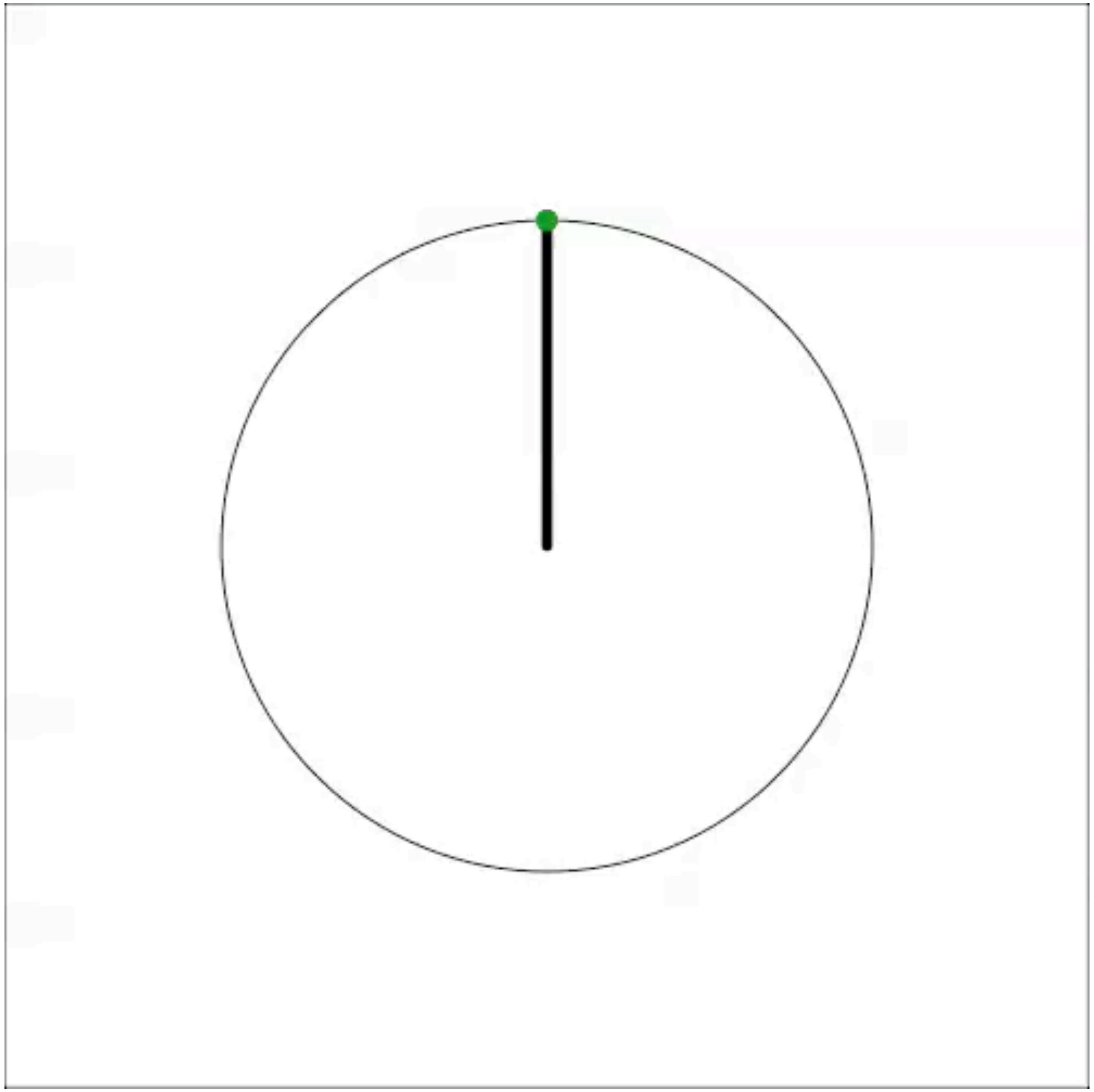
No fixed points for $K < K_c$

Unlocked



$K{=}.25K_c$

Locked



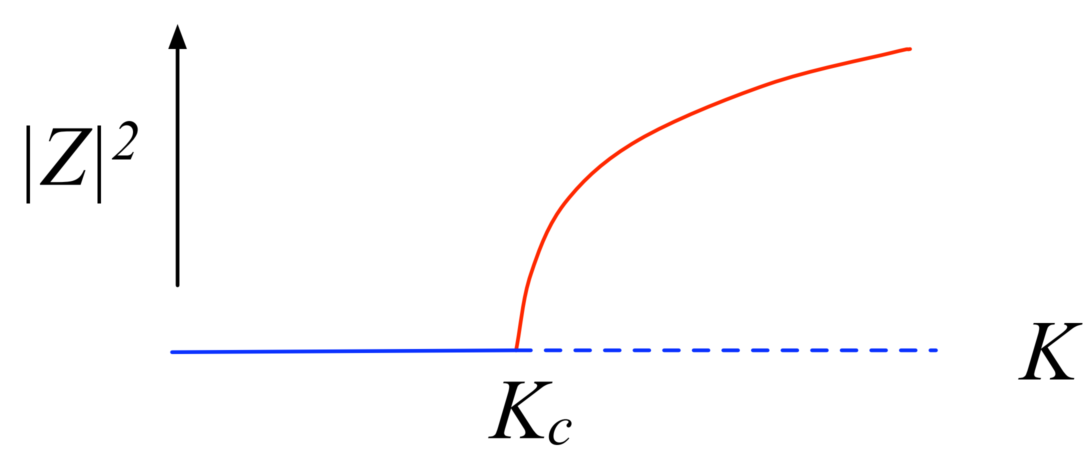
$K=2.5K_c$

Mean field theory

- Kuramoto showed in mean field theory i.e. $N \to \infty$, oscillators partially phase lock if $K > K_c = 2\gamma$ (width of frequency dist.)

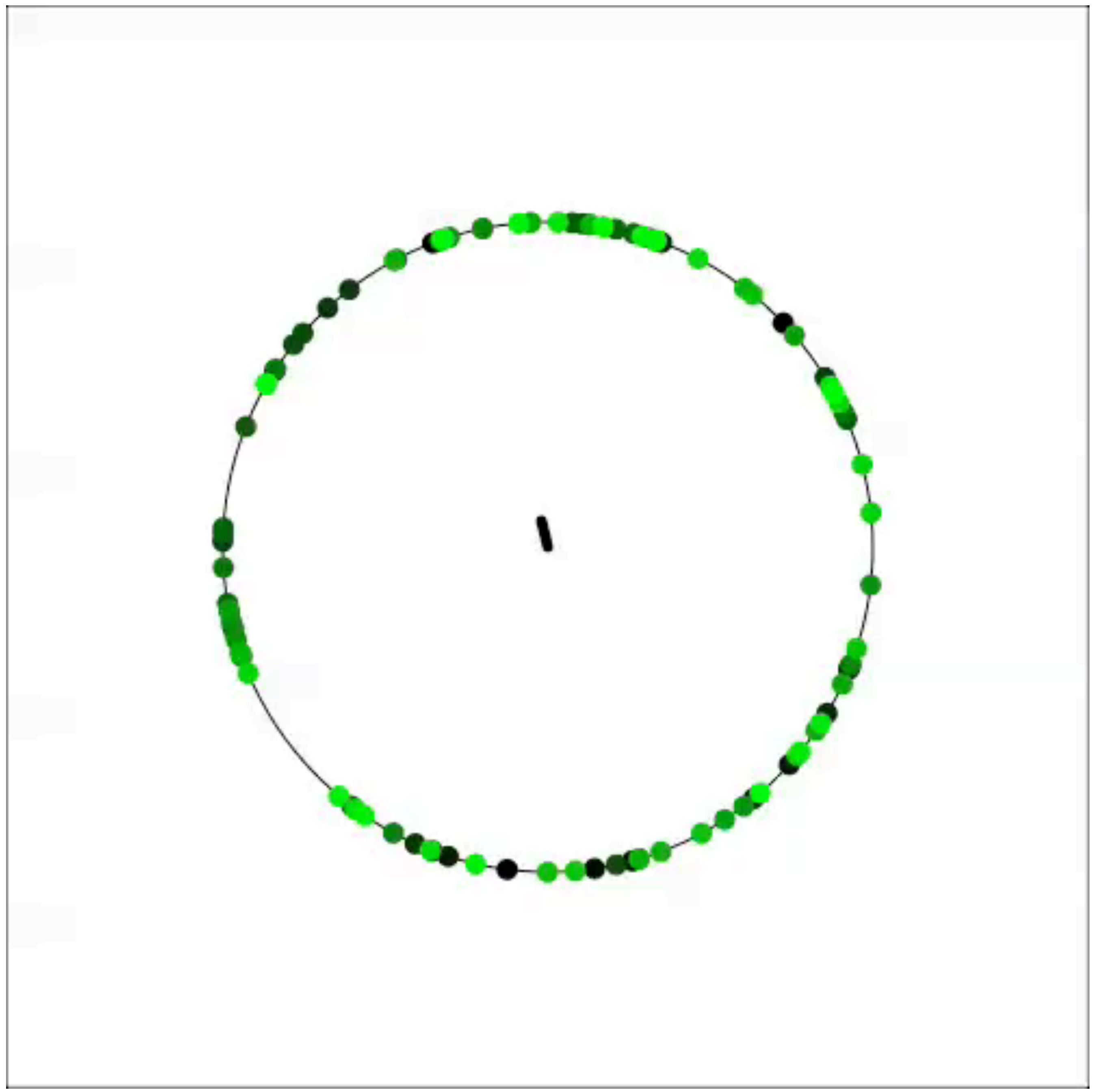
- Introduced order parameter

$$Z = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$$



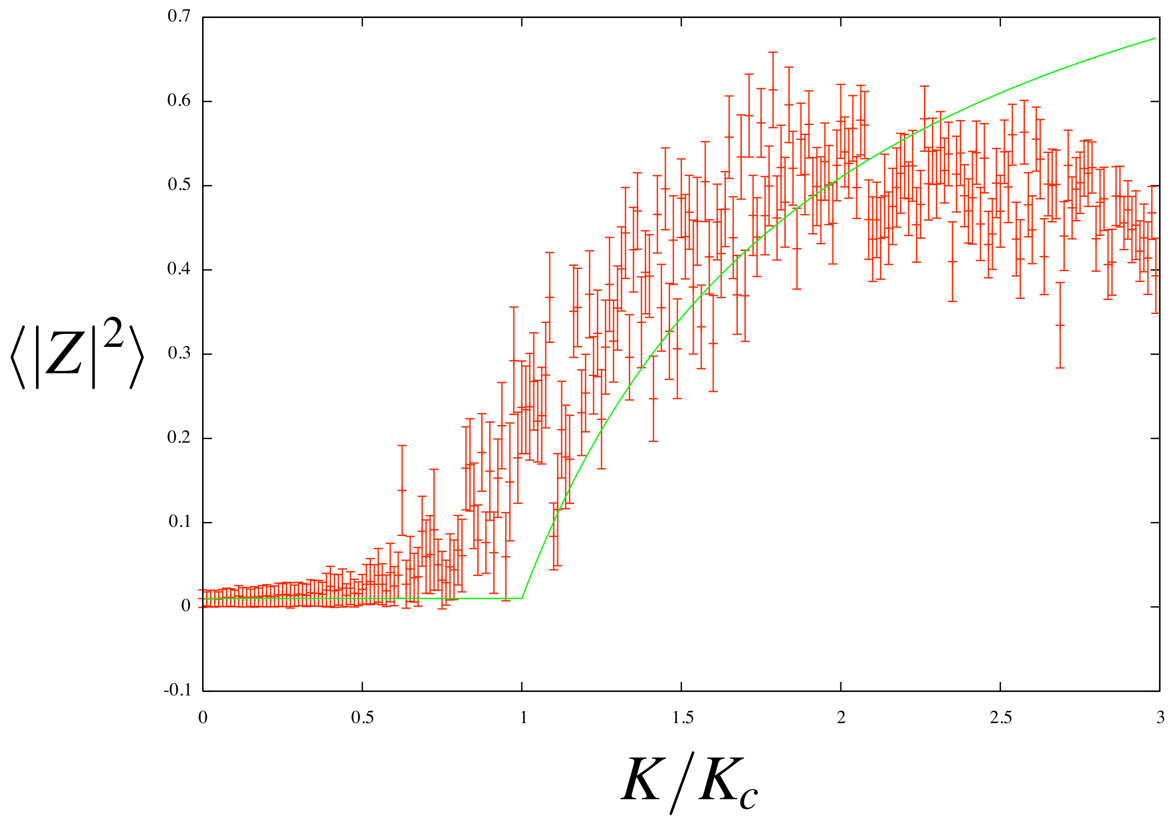
Example simulations

Subcritical (incoherent)



$N = 100, K = .5K_c$

Fluctuations due to finite size



Want to estimate finite-size fluctuations

Stability

- Kuramoto showed the existence of two branches (locked and incoherent) in 70's
- But did not calculate stability
- Postulated that incoherent state becomes unstable at critical point
- Stability of incoherent state calculated by Strogatz and Mirollo in 1990

Continuity equation approach

(Strogatz and Mirollo, 1990)

Probability density: $\rho(\theta, t)$

Oscillator dynamics: $$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N} f(\theta_j - \theta_i)$$

$$v \equiv \dot{\theta} = \omega + K \int_0^{2\pi} \int_{-\infty}^{\infty} f(\theta' - \theta)\rho(\theta', t) g(\omega') d\theta' d\omega'$$

oscillator “velocity”

Oscillator conservation implies:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (v\rho)}{\partial \theta} = 0$$

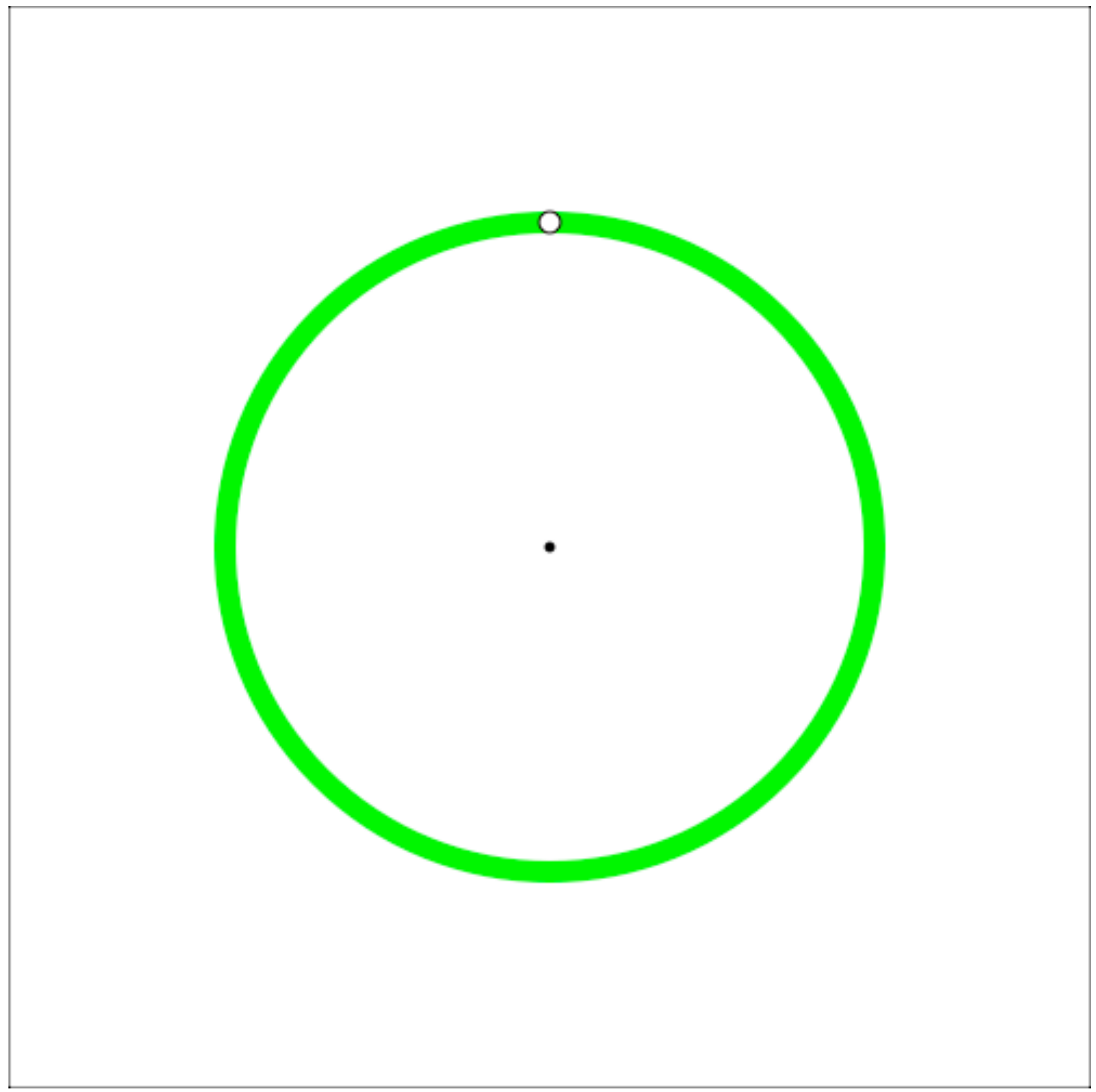
Continuity equation:

$$\frac{\partial \rho}{\partial t} + \omega \frac{\partial \rho}{\partial \theta} + K \frac{\partial}{\partial \theta} \int_{-\infty}^{\infty} \int_{0}^{2\pi} f(\theta' - \theta) \rho(\theta', \omega', t) \rho(\theta, \omega, t) d\theta' d\omega' = 0$$

Incoherent State

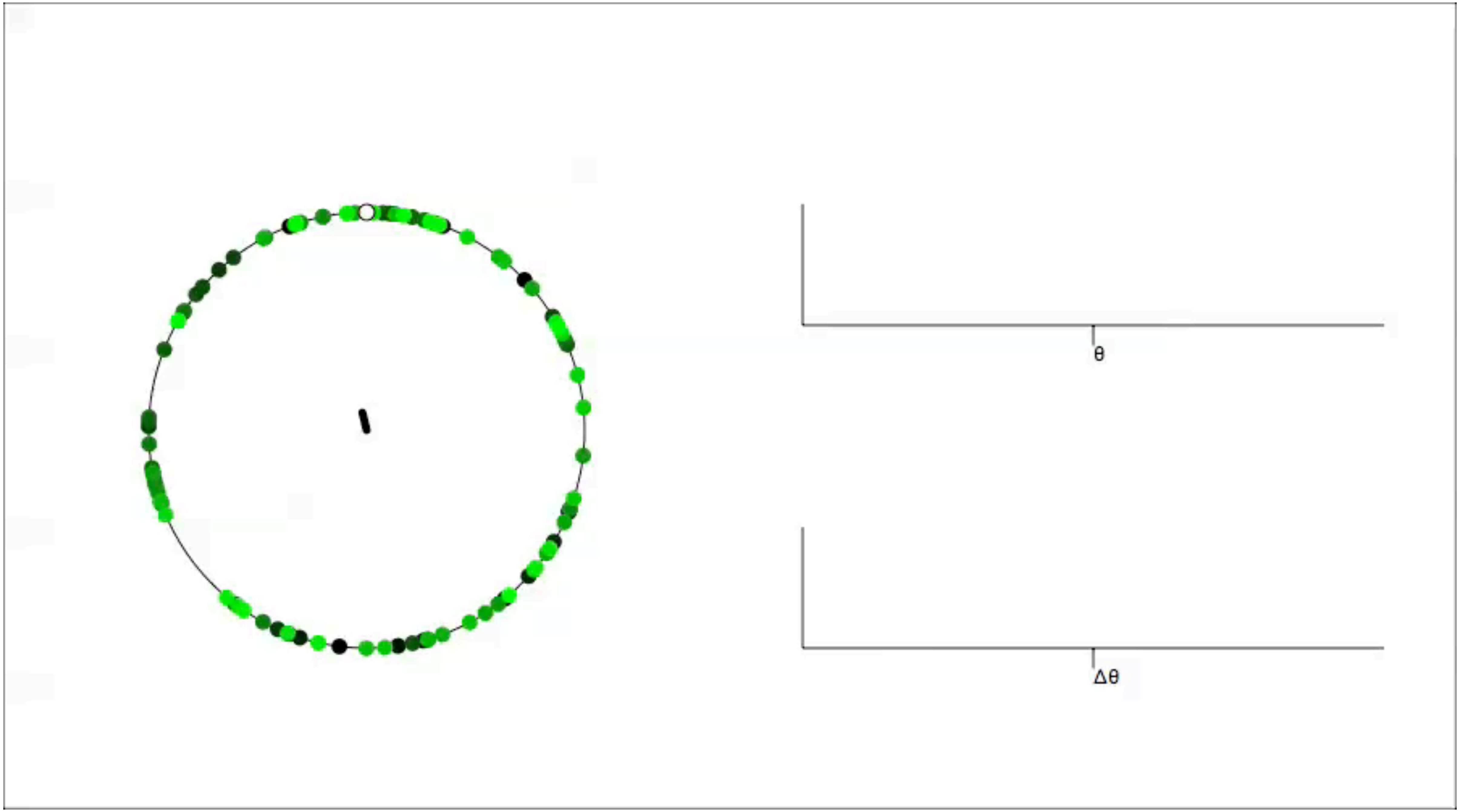
- Solution to continuity equation is the incoherent state $\rho(\theta) = 1/2\pi$
- Consider perturbations $\delta\rho e^{st}$, solve eigenvalue problem (spectrum)
- Unstable if s has positive real part
- Eigenvalues becomes unstable at K=K_c
- Continuous spectrum on imaginary axis
- Incoherent state is marginally stable!

A single oscillator in the incoherent state in the mean field limit $N \to \infty$



Oscillators don't interact - i.e. marginally stable modes

Finite Size effects



Recap

- Kuramoto model bifurcates from incoherence to partial locking at K_c
- Seen in order parameter Z
- Simulations with finite N show fluctuations
- Incoherent state is marginally stable in mean-field limit but seems stable in simulations
- Need to include finite N effects (same problem encountered in theory of gases and plasmas)

Kinetic theory approach

- Re-interpret the probability density

$$\eta(\theta,\omega,t) = \frac{1}{N}\sum_{i=1}^{N}\delta(\theta-\theta_i(t))\delta(\omega-\omega_i)$$

Continuity (“Klimontovich”) equation

$$\frac{\partial\eta}{\partial t} + \omega\frac{\partial\eta}{\partial\theta} + K\frac{\partial}{\partial\theta}\int_{-\infty}^{\infty}\int_{0}^{2\pi} f(\theta'-\theta)\eta(\theta',\omega',t)\eta(\theta,\omega,t)\,d\theta' d\omega' = 0$$

includes finite size fluctuations

- Klimontovich equation contains all the information of the original system (density is not smooth)
- To be useful, need to do averaging
- Strogatz and Mirollo considered the "mean field limit" where the density is smooth
- To consider finite size fluctuations, need to use the Klimontovich equation

Items to calculate

- Fluctuations = Variance of order parameter (Two-oscillator density function)

$$\langle |Z|^2 \rangle = \int d\omega d\omega' d\theta d\theta' \langle \eta(\omega,\theta,t)\eta(\omega',\theta',t)\rangle e^{i(\theta-\theta')}$$

- Stability = Spectrum of linearized Klimontovich equation (to order *1/N*)
- Estimate using moment hierarchy or probability density functional of η (field theory)

Moment hierarchy

Take ensemble average over initial condition and frequencies of Klimontovich equation

$$\frac{\partial\eta}{\partial t}+\omega\frac{\partial\eta}{\partial\theta}+K\frac{\partial}{\partial\theta}\int_{-\infty}^{\infty}\int_{0}^{2\pi}f(\theta'-\theta)\eta(\theta',\omega',t)\eta(\theta,\omega,t)d\theta' d\omega'=0$$

One oscillator density $\rho(\theta,\omega)=\langle\eta(\theta,\omega)\rangle$ satisfies

$$\frac{\partial\rho}{\partial t}+\omega\frac{\partial\rho}{\partial\theta}+K\frac{\partial}{\partial\theta}\int_{-\infty}^{\infty}\int_{0}^{2\pi}f(\theta'-\theta)\rho(x,t)\rho(x',t)d\theta' d\omega'=-K\frac{\partial}{\partial\theta}\int_{-\infty}^{\infty}\int_{0}^{2\pi}f(\theta'-\theta)C(x,x',t)d\theta' d\omega'$$

where $x=(\theta,\omega)$

$$C(x,x',t)=\langle\eta(x,t)\eta(x',t)\rangle-\rho(x,t)\rho(x',t)+\frac{1}{N}\rho(x,t)\rho(x',t)-\frac{1}{N}\delta(x-x')\rho(x',t)$$

- One-oscillator density depends on two-oscillator density (correlation function)

- Form two-oscillator density equation: multiply Klimontovich equation by η and take ensemble average

- Two-oscillator density depends on three-oscillator density, ...

- BBGKY moment hierarchy

- Must truncate to be useful

First two equations of hierarchy (with Gaussian closure)

$$\frac{\partial \rho}{\partial t} + \omega \frac{\partial \rho}{\partial \theta} + K \frac{\partial}{\partial \theta} \int_{-\infty}^{\infty} \int_{0}^{2\pi} f(\theta' - \theta) \rho(x,t) \rho(x',t) d\theta' d\omega' = -K \frac{\partial}{\partial \theta} \int_{-\infty}^{\infty} \int_{0}^{2\pi} f(\theta' - \theta) C(x,x',t) d\theta' d\omega'$$

$$\{\frac{\partial}{\partial t} + \omega_1 \frac{\partial}{\partial \theta_1} + \omega_2 \frac{\partial}{\partial \theta_2} + K \int_{-\infty}^{\infty} \int_{0}^{2\pi} [\frac{\partial}{\partial \theta_1} f(\theta_3 - \theta_1) + \frac{\partial}{\partial \theta_2} f(\theta_3 - \theta_2)] \rho(x_3,t) d\theta_3 d\omega_3 \} C(x_1,x_2,t)$$

$$+K \int_{-\infty}^{\infty} \int_{0}^{2\pi} \frac{\partial}{\partial \theta_1} f(\theta_3 - \theta_1) \rho(x_1,t) C(x_2,x_3,t) d\theta_3 d\omega_3$$

$$+K \int_{-\infty}^{\infty} \int_{0}^{2\pi} \frac{\partial}{\partial \theta_2} f(\theta_3 - \theta_2) \rho(x_2,t) C(x_3,x_1,t) d\theta_3 d\omega_3 \}$$

$$= -\frac{K}{N} [\frac{\partial}{\partial \theta_1} f(\theta_2 - \theta_1) + \frac{\partial}{\partial \theta_2} f(\theta_1 - \theta_2)] \rho(x_1,t) \rho(x_2,t)$$

Captures two-oscillator interactions (finite-size fluctuations) but unwieldy to solve

If we ignore two-oscillator interactions (collisions)

get "Vlasov" equation

$$\frac{\partial\rho}{\partial t}+\omega\frac{\partial\rho}{\partial\theta}+K\frac{\partial}{\partial\theta}\int_{-\infty}^{\infty}\int_{0}^{2\pi}f(\theta'-\theta)\rho(x,t)\rho(x',t)d\theta' d\omega'=0$$

this looks just like Klimontovich equation

$$\frac{\partial\eta}{\partial t}+\omega\frac{\partial\eta}{\partial\theta}+K\frac{\partial}{\partial\theta}\int_{-\infty}^{\infty}\int_{0}^{2\pi}f(\theta'-\theta)\eta(x,t)\eta(x,t)d\theta' d\omega'=0$$

Difference is the smoothness of solutions

Most people go straight to Vlasov equation but it is only valid in the mean field limit

Statistical field theory

Klimontovich equation:

$$\mathcal{O}[\eta] \equiv \frac{\partial \eta}{\partial t} + \omega \frac{\partial \eta}{\partial \theta} + K \frac{\partial}{\partial \theta} \int_{-\infty}^{\infty} \int_{0}^{2\pi} f(\theta' - \theta) \eta(\theta', \omega', t) \eta(\theta, \omega, t) d\theta' d\omega' = \delta(t - t_0) \eta_0(\theta, \omega)$$

Equation for a probability density η

Want to calculate moments of the density η

Need a "density of the density"

Statistical field theory

$$\mathcal{O}[\eta] \equiv \frac{\partial \eta}{\partial t} + \omega \frac{\partial \eta}{\partial \theta} + K \frac{\partial}{\partial \theta} \int_{-\infty}^{\infty} \int_{0}^{2\pi} f(\theta' - \theta) \eta(\theta', \omega', t) \eta(\theta, \omega, t) d\theta' d\omega' = \delta(t - t_0) \eta_0(\theta, \omega)$$

Probability density functional of the density is

$$\mathcal{F}[\eta, \eta_0] = \delta \left[N \left\{ \mathcal{O}[\eta(\theta, \omega, t)] - \delta(t - t_0) \eta_0(\theta, \omega) \right\} \right]$$

Marginalize over initial densities

$$\mathcal{F}[\eta(\theta, \omega, t)] = \int \mathcal{D}\eta_0 \mathcal{F}_0[\eta_0] \delta \left[N \left\{ \mathcal{O}[\eta(\theta, \omega, t)] - \delta(t - t_0) \eta_0(\theta, \omega) \right\} \right]$$

Path integral over initial densities

Functional Fourier decomposition

$$\delta(x) \propto \int e^{-ikx} dk$$

$$\mathcal{F}[\eta(\theta,\omega,t)] = \int \mathcal{D}\tilde{\eta}\mathcal{D}\eta_0 \mathcal{F}_0[\eta_0] \exp\left(-N \int d\theta d\omega dt\, \tilde{\eta}\left[\mathcal{O}[\eta] - \delta(t-t_0)\eta_0(\theta,\omega)\right]\right)$$

where $\tilde{\eta}(\theta,\omega,t)$ is the "MSR response field"

Convenient to consider joint density functional

$$\tilde{\mathcal{F}}[\eta,\tilde{\eta}] = \int \mathcal{D}\eta_0 \mathcal{F}_0[\eta_0] \exp\left(-N \int d\theta d\omega dt\, \tilde{\eta}\mathcal{O}[\eta] + N \int d\theta d\omega \tilde{\eta}(\theta,\omega,t_0)\eta_0(\theta,\omega)\right)$$

which obeys $$1 = \int \mathcal{D}\eta\mathcal{D}\tilde{\eta}\tilde{\mathcal{F}}[\eta,\tilde{\eta};\eta_0]$$

Integrate over initial data and frequencies

$$\tilde{\mathcal{F}}[\eta,\tilde{\eta}] = \exp\left(-N\int d\theta d\omega dt\, \tilde{\eta}\mathcal{O}[\eta] + N\ln\left\{1+\int d\theta d\omega\left[e^{\tilde{\eta}(\theta,\omega,t_0)}-1\right]\rho_0(\theta,\omega)\right\}\right)$$

To simplify initial condition contribution

Make the following transformation

$$\varphi(\theta,\omega,t) = \eta\exp(-\tilde{\eta})$$

$$\tilde{\varphi}(\theta,\omega,t)+1 = \exp(\tilde{\eta})$$

Moments of η can be expressed in terms of moments of φ

Results in

$$\tilde{\mathcal{F}}[\varphi, \tilde{\varphi}] = \exp\left(-NS[\varphi, \tilde{\varphi}]\right)$$

with "action"

$$S[\varphi,\tilde{\varphi}] = \int d\omega d\theta dt \left[\tilde{\varphi}\left(\frac{\partial}{\partial t} + \omega\frac{\partial}{\partial\theta}\right)\varphi + K\int d\omega' d\theta' \left(\tilde{\varphi}'\tilde{\varphi} + \tilde{\varphi}\right)\frac{\partial}{\partial\theta}\left\{f(\theta'-\theta)\varphi'\varphi\right\}\right]$$

$$-\ln\left[1 + \int d\theta d\omega \tilde{\varphi}(\theta,\omega,t_0)\rho_0(\theta,\omega)\right]$$

Calculate moments perturbatively using method of steepest descents (loop expansion)

Loop Expansion

Moments $$\langle \varphi^n \tilde{\varphi}^m \rangle = \int D\varphi D\tilde{\varphi}\ \varphi^n \tilde{\varphi}^m\ e^{-NS[\varphi,\tilde{\varphi}]}$$

- Steepest descent asymptotic expansion (e.g. expand around saddle point)
- Expand path integral in terms of “Gaussian moments” in powers of *1/N*
- Can keep track of terms by using “Feynman diagrams”

Simple example

$$\langle u^2 \rangle = \int_{-\infty}^{\infty} u^2 e^{N(iP^{-1}uv - av^2 + ibu^2v - cu^2v^2)} d\omega \qquad d\omega = \frac{N}{2\pi P} du dv$$

$$\sim \int_{-\infty}^{\infty} u^2 e^{iNP^{-1}uv} (1 - aNv^2 + ibNu^2v - cNu^2v^2 + \cdots) d\omega$$

Use identity $$\int_{-\infty}^{\infty} u^n v^m e^{iNP^{-1}uv}\, d\omega = n! \frac{i^n}{N^n} P^n \delta_{nm}$$

$$\langle u^2 \rangle \sim \frac{2}{N} aP^2 + \frac{24}{N^2} acP^4 + \cdots$$

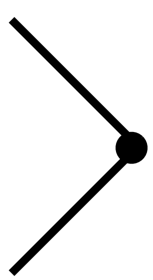
Only terms with equal numbers of v and u survive

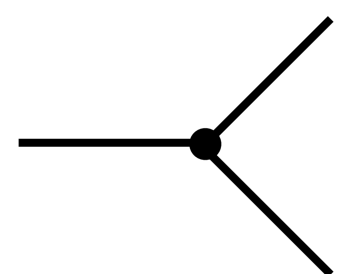
Diagrams

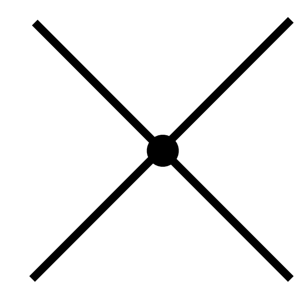
Bookkeeping for expansion can be aided with diagrams with rules for assembly

Propagator P/N 

Vertices

$-Na$ 

iNb 

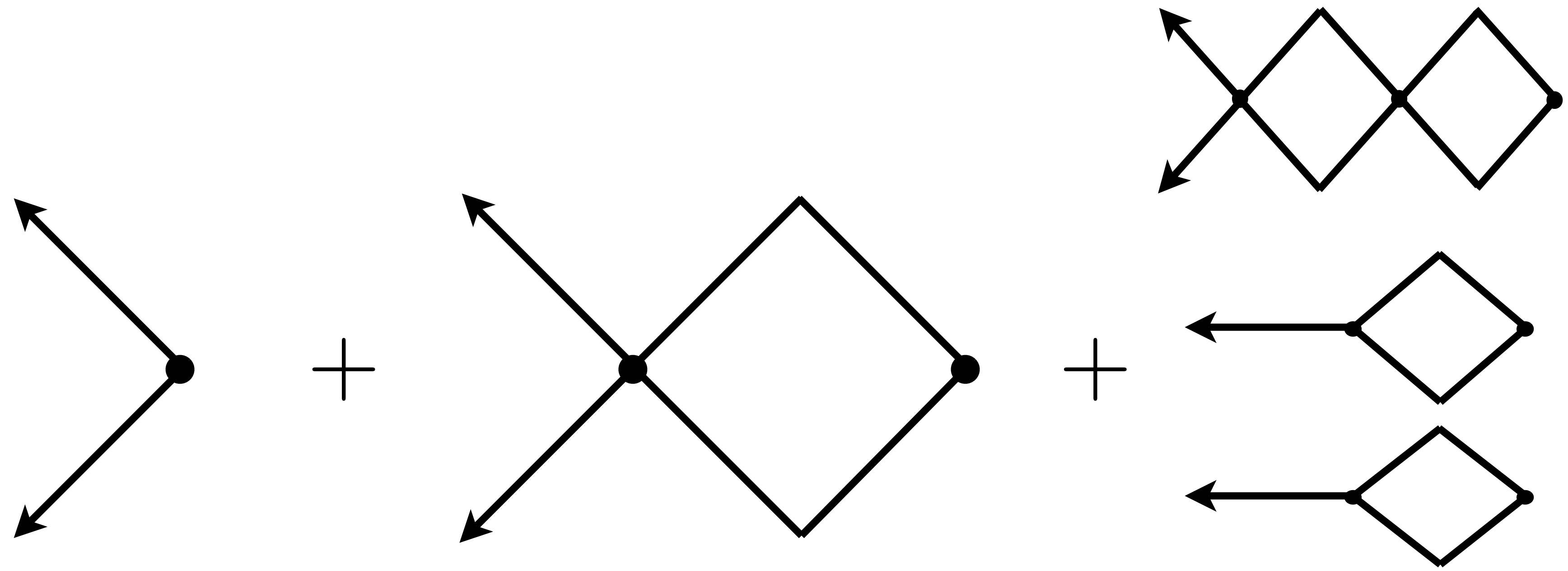
$-Nc$ 

Moment $\langle u^n v^m \rangle$ is the sum of all diagrams with n outgoing legs and m incoming legs

Legs of vertices are joined by propagators

Loop expansion to order *1/N²*

$$\langle u^2 \rangle =$$



$$= \quad 2Na(P/N)^2 \quad + \quad 24NaNc(P/N)^4 \quad + \quad O(1/N^3)$$

Combinatorics is only tricky part

Field Theory for Kuramoto

$$\langle \varphi^n \tilde{\varphi}^m \rangle = \int D\varphi D\tilde{\varphi}\ \varphi^n \tilde{\varphi}^m\ e^{-NS[\varphi,\tilde{\varphi}]}$$

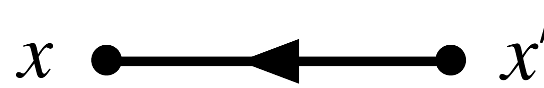
$$S[\varphi,\tilde{\varphi}] = S_F[\varphi,\tilde{\varphi}] + S_I[\varphi,\tilde{\varphi}]$$

$$S_F[\varphi,\tilde{\varphi}] = \int d\omega d\theta dt \tilde{\varphi} \left[\left(\frac{\partial}{\partial t} + \omega \frac{\partial}{\partial \theta} \right) \varphi + K \int d\omega' d\theta' \frac{\partial}{\partial \theta} \left\{ f(\theta' - \theta) \left(\varphi' \rho + \rho' \varphi \right) \right\} \right] \equiv \tilde{\varphi} \cdot N^{-1} P_0^{-1} \cdot \varphi$$

$$S_I[\varphi,\tilde{\varphi}] = \int d\omega d\theta dt \tilde{\varphi} \left[K \int d\omega' d\theta' \frac{\partial}{\partial \theta} \left\{ f(\theta' - \theta) \left(\varphi' \varphi \right) \right\} + K \int d\omega' d\theta' \tilde{\varphi}' \frac{\partial}{\partial \theta} \left\{ f(\theta' - \theta) \left(\varphi' + \rho' \right) \left(\varphi + \rho \right) \right\} \right]$$

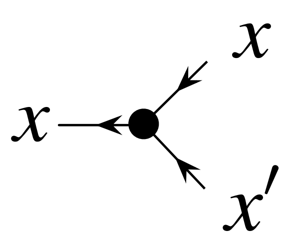
$$- \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k} \left[\int d\theta d\omega \tilde{\varphi}(\theta,\omega,t_0) \rho_0(\theta,\omega) \right]^k$$

Diagrams for Kuramoto model



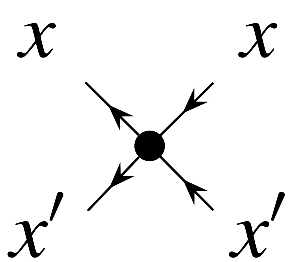
$$P_0(x,t|x',t')$$

I



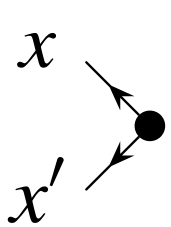
$$KN\frac{\partial}{\partial\theta}\{f(\theta'-\theta)\dots\}$$

II



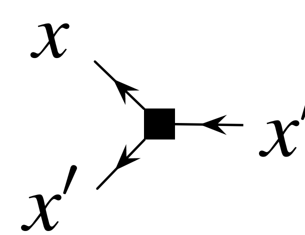
$$KN\frac{\partial}{\partial\theta}\{f(\theta'-\theta)\dots\}$$

III



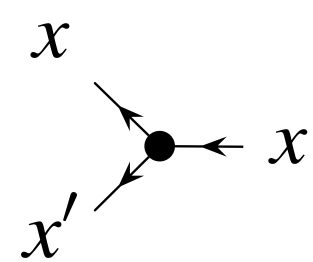
$$\frac{KN}{(2\pi)^2}g(\omega)g(\omega')\frac{\partial}{\partial\theta}\{f(\theta'-\theta)\}$$

IV



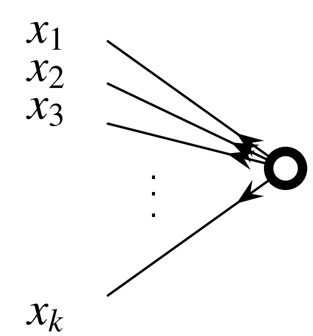
$$\frac{KN}{2\pi}g(\omega)\frac{\partial}{\partial\theta}\{f(\theta'-\theta)\}$$

V

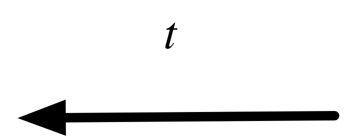


$$\frac{KN}{2\pi}g(\omega')\frac{\partial}{\partial\theta}\{f(\theta'-\theta)\dots\}$$

VI



$$\frac{(-1)^{k+1}}{k}\prod_{j=2}^{k}\rho_0(\theta_j,\omega_j)$$



Integrate over *x*, *x'* in the diagrams

Finite size fluctuations

Variance of order parameter

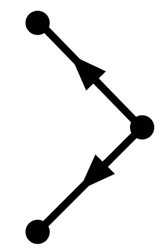
$$\langle |Z|^2 \rangle = \int d\omega d\omega' d\theta d\theta' C(\omega, \theta; \omega', \theta', t) e^{i(\theta-\theta')} + \frac{1}{N}$$

$$C(x_1, t_1; x_2, t_2) = \langle \varphi(x_1, t_1) \varphi(x_2, t_2) \rangle$$

Use loop expansion to compute correlation function

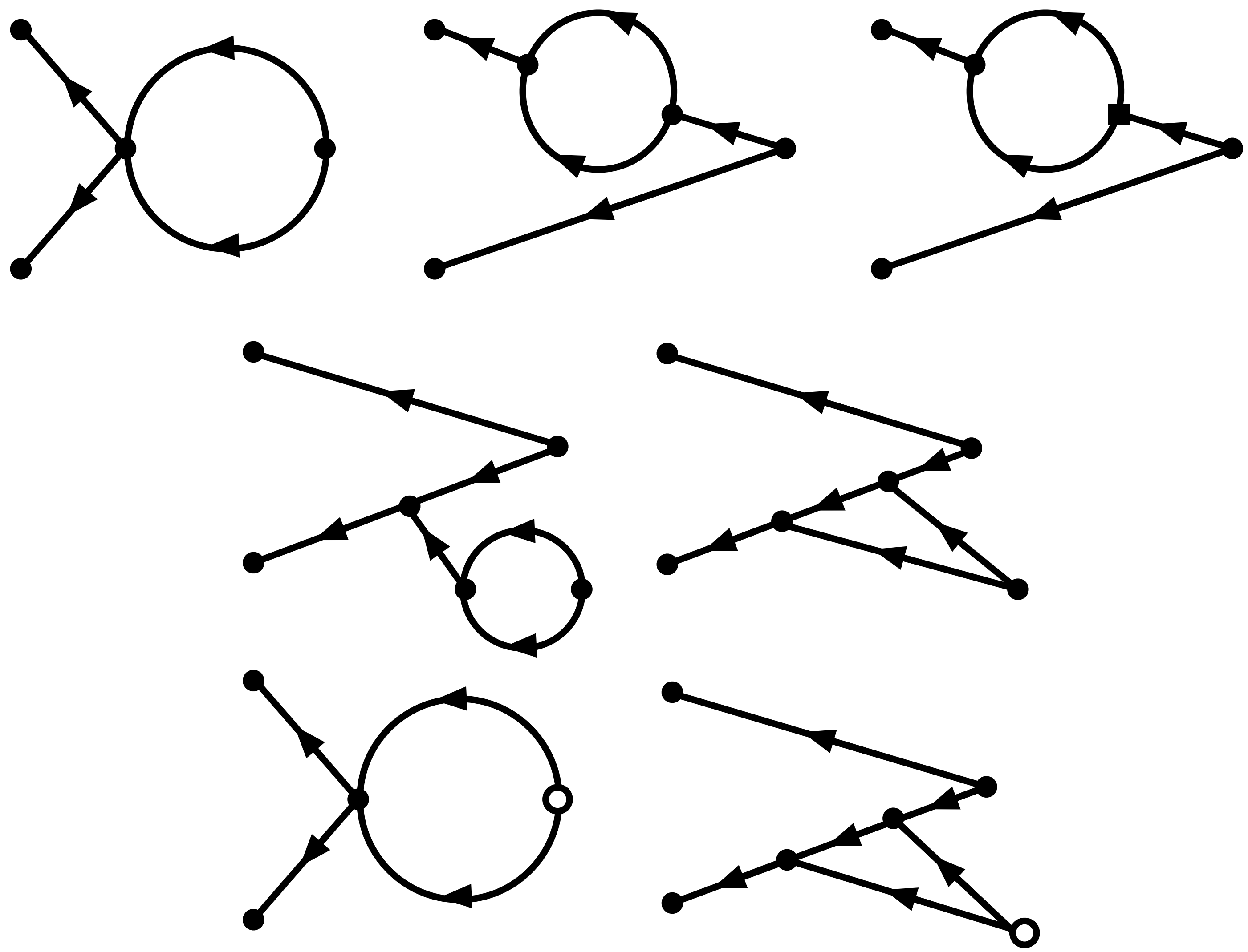
Diagrams of correlation function

a)



Tree level

b)



One loop

Order *1/N* expansion (tree level)

$$C(x_1,t_1;x_2,t_2) = -\frac{K}{N}\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} d\omega_1' d\omega_2' \int_0^{2\pi} d\theta_1' d\theta_2' \int_{t_0}^{t} dt' P_0(x_1,x_1',t_1-t') P_0(x_2,x_2',t_2-t')$$

$$\times\left[\frac{\partial}{\partial\theta_1'} f(\theta_2'-\theta_1') + \frac{\partial}{\partial\theta_2'} f(\theta_1'-\theta_2')\right] g(\omega_1') g(\omega_2')$$

Propagator $P_0(\theta,\omega,t|\theta',\omega',t') = \langle\varphi(\theta,\omega,t)\tilde{\varphi}(\theta',\omega',t')\rangle$ satisfies

$$\left[\frac{\partial}{\partial t} + \omega\frac{\partial}{\partial\theta}\right] P_0(x,x',t-t') + K\frac{g(\omega)}{2\pi}\frac{\partial}{\partial\theta}\int_{-\infty}^{\infty}\int_0^{2\pi} f(\theta_1-\theta) P_0(x_1,x',t-t') d\theta_1 d\omega_1$$

$$= \frac{1}{N}\delta(\theta-\theta')\delta(\omega-\omega')\delta(t-t')$$

Laplace-Fourier transform in t and θ:

$$[s+in\omega]\,\tilde{P}_0(n,\omega,\omega',s)+inKg(\omega)f(-n)\int_{-\infty}^{\infty}\tilde{P}_0(n,\omega_1,\omega',s)d\omega_1=\frac{1}{2\pi N}\delta(\omega-\omega')$$

Integrate over ω:

$$\int d\omega\tilde{P}_0(n,\omega,\omega',s)=\frac{1}{2\pi N}\frac{1}{s+in\omega'}\frac{1}{\Lambda_n(s)}$$

$$\Lambda_n(s)=1+inKf(-n)\int d\omega\frac{g(\omega)}{s+in\omega}$$

"Dielectric function"

Tree level propagator:

$$\tilde{P}_0(n,\omega,\omega',s)=\frac{1}{2\pi N}\frac{\delta(\omega-\omega')}{s+in\omega}-\frac{1}{2\pi N}\frac{inKg(\omega)f(-n)}{(s+in\omega)\,(s+in\omega')}\frac{1}{\Lambda_n(s)}$$

Fluctuations

$$\langle |Z|^2(\tau)\rangle = \int d\omega d\omega' d\theta d\theta' C(x,x',\tau) e^{i(\theta-\theta')} + \frac{1}{N}$$

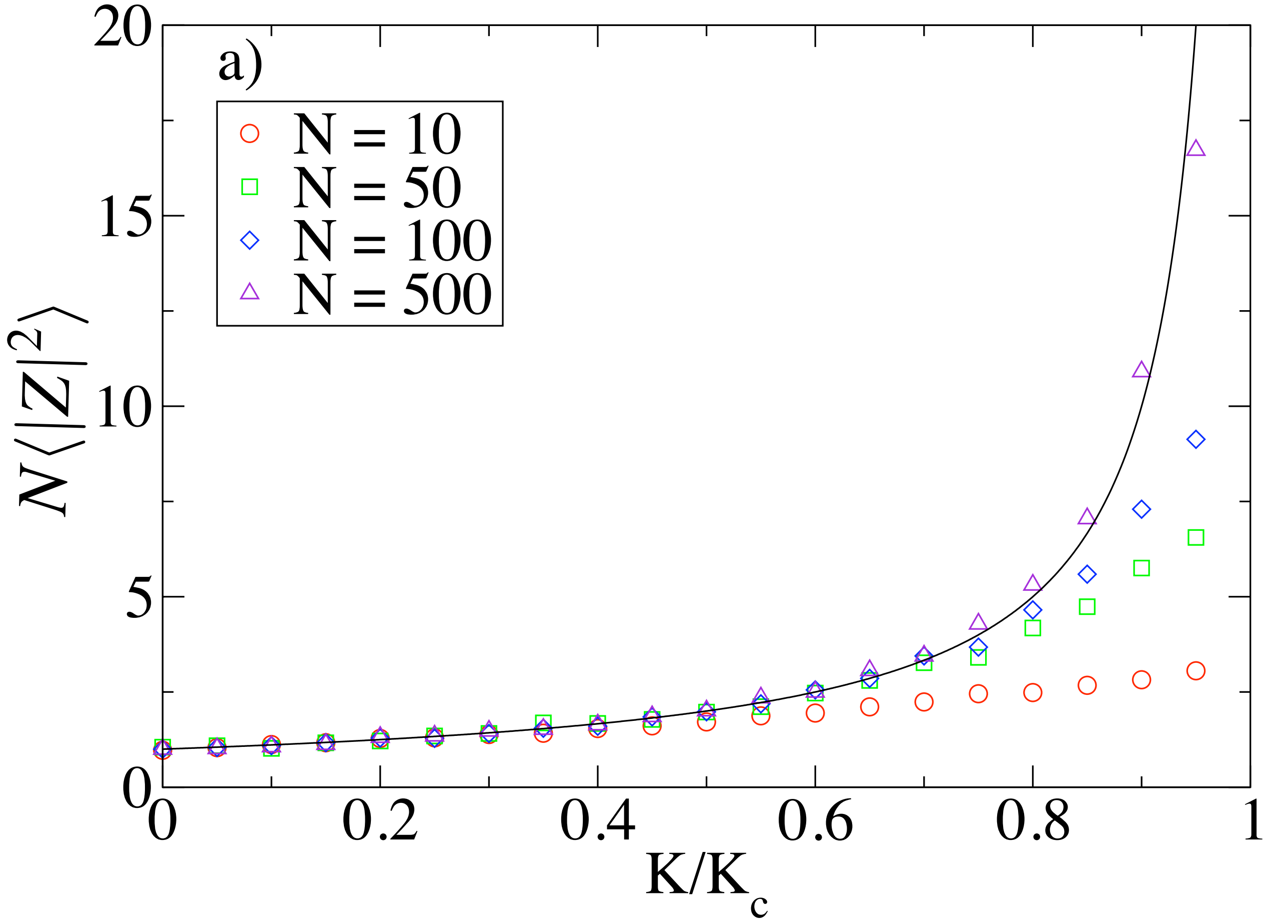
$$\langle |Z|^2(\tau)\rangle = \frac{2}{iKN\pi}\int_{\mathcal{L}} ds \frac{\Lambda_1(s-s_0)-1}{\Lambda_1(s-s_0)} \mathrm{Res}\left[\frac{-1}{\Lambda_1(s)}\right]_{s=s_0} \frac{1}{s} e^{s\tau} + \frac{1}{N}$$

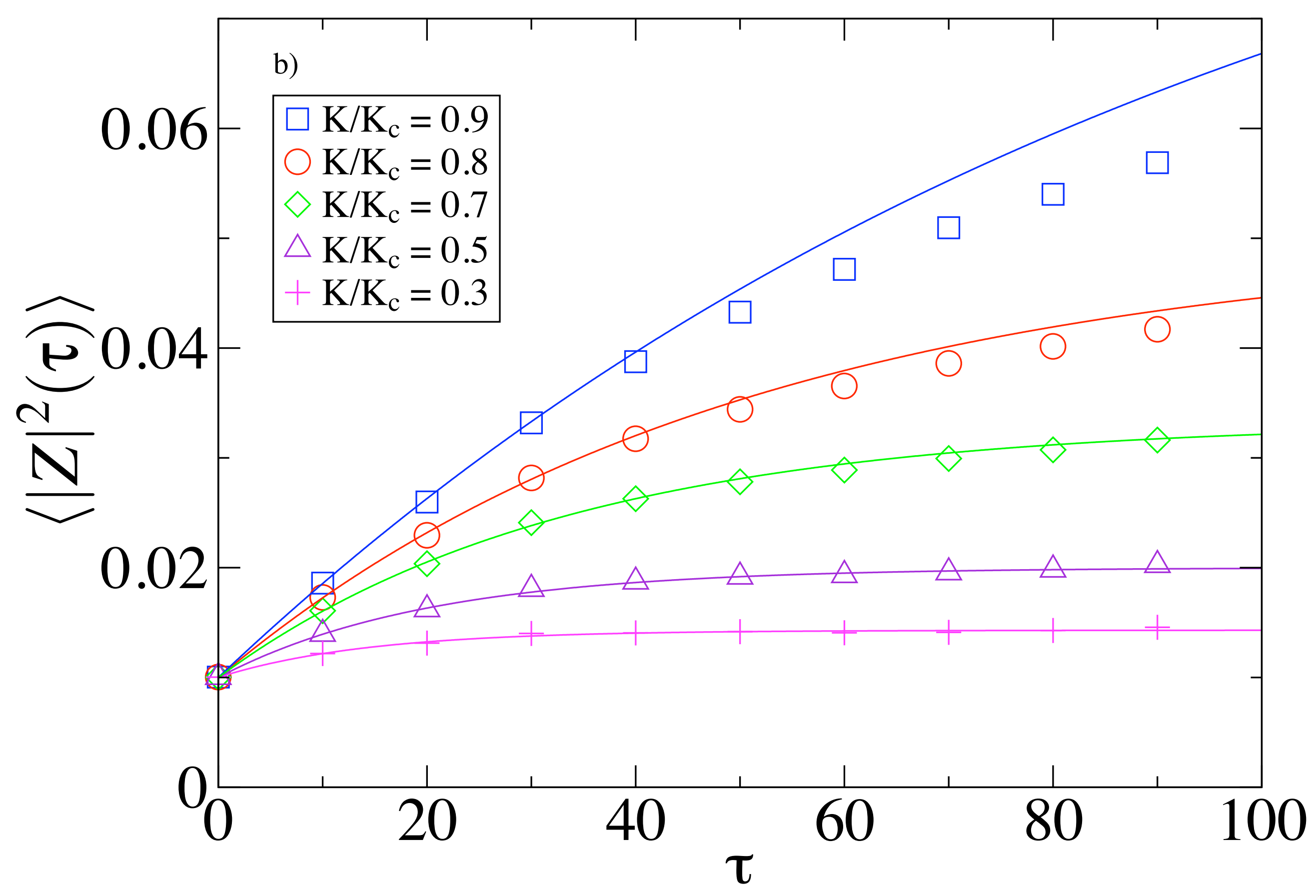
for $$g(\omega) = \frac{1}{\pi}\frac{\gamma}{\omega^2+\gamma^2}$$

$$\langle |Z|^2(\tau)\rangle = \frac{1}{N}\frac{K_c}{K_c-K} - \frac{1}{N}\frac{K}{K_c-K} e^{-(K_c-K)\tau}$$

Expect theory to breakdown near critical point

Comparison to simulations





Hildebrand, Buice and Chow, PRL (2007)

Stability of incoherent state mean field theory

Consider perturbations to Vlasov equation

Sub $\rho = g(\omega)/2\pi + \delta\rho$ in

$$\frac{\partial\rho}{\partial t} + \omega\frac{\partial\rho}{\partial\theta} + K\frac{\partial}{\partial\theta}\int_{-\infty}^{\infty}\int_{0}^{2\pi} f(\theta' - \theta)\rho(\theta', \omega', t)\rho(\theta, \omega, t)\, d\theta' d\omega' = 0$$

To obtain

$$\frac{\partial}{\partial t}\delta\rho = -\omega\frac{\partial}{\partial\theta}\delta\rho - K\frac{g(\omega)}{2\pi}\frac{\partial}{\partial\theta}\int_{-\infty}^{\infty}\int_{0}^{2\pi} f(\theta_1 - \theta)\delta\rho(x_1, x', t - t')\, d\theta_1 d\omega_1 \equiv L \cdot \delta\rho$$

- Compute spectrum of linear operator L
i.e. values of s where $(s\text{-}L)^{-1}$ is unbounded

- Incoherent state stable if spectrum in left plane

- Spectrum given by poles of propagator (Green's function)

$$\left[\frac{\partial}{\partial t} + \omega\frac{\partial}{\partial\theta}\right] P_0(x,x',t-t') + K\frac{g(\omega)}{2\pi}\frac{\partial}{\partial\theta}\int_{-\infty}^{\infty}\int_0^{2\pi} f(\theta_1-\theta)P_0(x_1,x',t-t')d\theta_1 d\omega_1$$

$$= \frac{1}{N}\delta(\theta-\theta')\delta(\omega-\omega')\delta(t-t')$$

- In Fourier-Laplace space

$$\tilde{P}_0(n,\omega,\omega',s) = \frac{1}{2\pi N}\frac{\delta(\omega-\omega')}{s+in\omega} - \frac{1}{2\pi N}\frac{inKg(\omega)f(-n)}{(s+in\omega)\,(s+in\omega')}\frac{1}{\Lambda_n(s)}$$

Spectrum

- Continuous spectrum on imaginary axis (marginally stable modes)
- Point spectrum (eigenvalues) given by zeros of dielectric function (analytically continue)
- One eigenvalue in left plane, crosses imaginary axis at critical point $K=K_c$
- Incoherent state is marginally stable in mean field limit (tree level)

Order parameter dynamics near incoherent state

$$Z(t) = \frac{1}{N}\sum_j e^{i\theta_j} = \int d\theta d\omega \eta(\theta,\omega,t) e^{i\theta}$$

$$\langle \delta Z(t) \rangle = \int d\theta d\omega \delta\rho(\theta,\omega,t) e^{i\theta}$$

For smooth perturbation $\delta\rho_0(\theta,\omega) = c(\theta)g(\omega)$

$$\delta\rho(\theta,\omega,t) = N\int_{-\infty}^{\infty}\int_0^{2\pi} P_0(x,x',t)c(\theta')g(\omega')d\theta' d\omega'$$

In Laplace space

$$\langle \delta \tilde{Z}(s) \rangle = c_{-1} \frac{\Lambda_{-1}(s) - 1}{K/2} \frac{1}{\Lambda_{-1}(s)}$$

For Cauchy

$$g(\omega) = \frac{1}{\pi} \frac{\gamma}{\gamma^2 + \omega^2}$$

$$\Lambda_{\pm 1}(s) = \frac{s + \gamma - \frac{K}{2}}{s + \gamma}$$

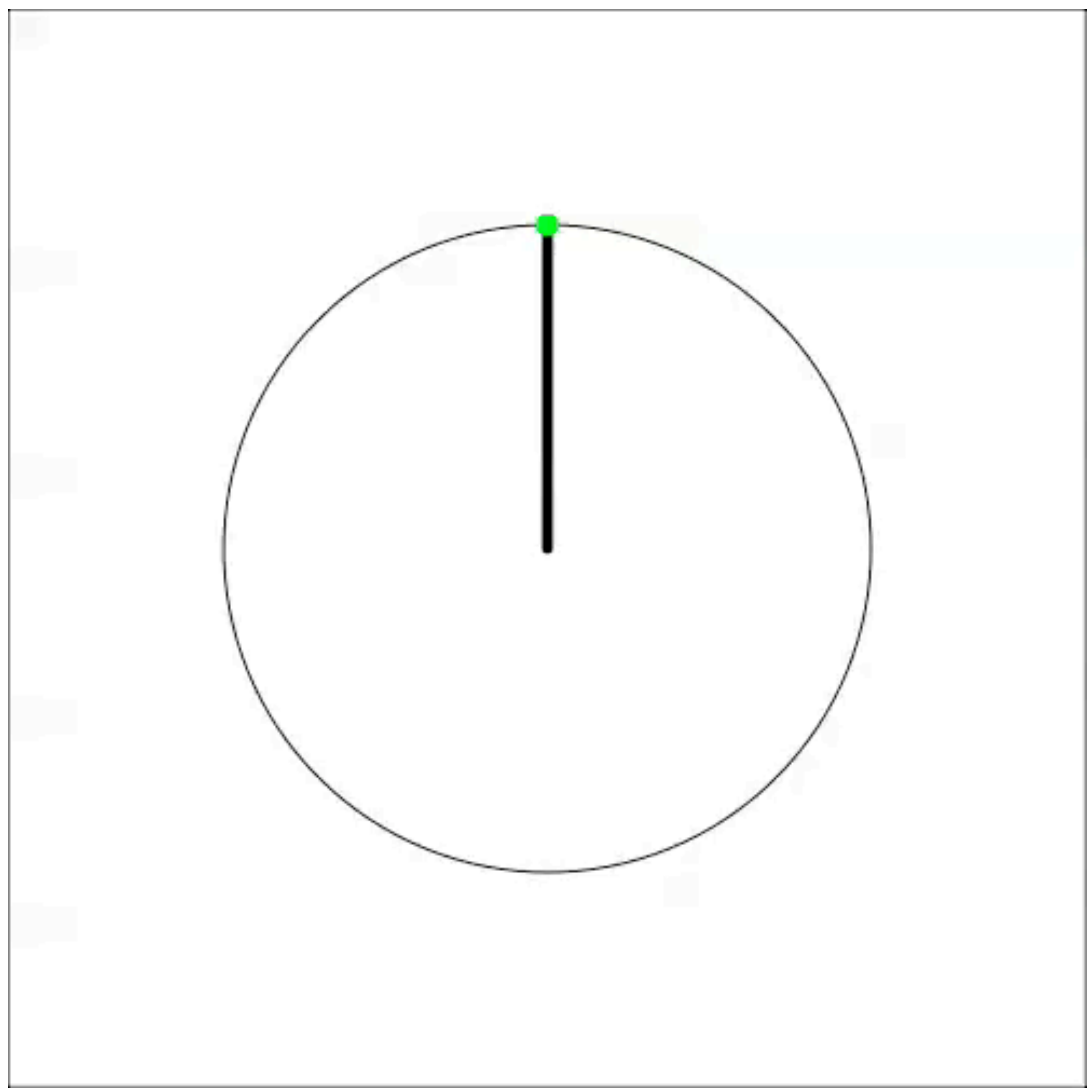
$$K_c = 2\gamma$$

Order parameter decays to zero by dephasing

$$\langle \delta Z \rangle = c_{-1} e^{-\left(\gamma - \frac{K}{2}\right)\tau}$$

Dephasing for smooth perturbation

- Strogatz, Mirollo and Matthews (1992) showed that dephasing leads to effective dissipation of Z (like Landau damping in plasma physics)



Non-smooth perturbation

Fix a single oscillator

$$\delta\rho_0(\theta,\omega) = \frac{1}{N}\left[-\frac{g(\omega)}{2\pi} + \delta(\theta-\theta_0)\delta(\omega-\omega_0)\right]$$

$$\langle\tilde{Z}(s)\rangle = \frac{1}{N}\frac{1}{s-in\omega_0}\frac{1}{\Lambda_{-1}(s)}$$

Z oscillates

$$\langle Z(t)\rangle = \frac{e^{i\theta_0}}{N}\frac{1}{\omega_0^2+\left(\gamma-\frac{K}{2}\right)^2}\left[\left(\gamma\left(\gamma-\frac{K}{2}\right)+\omega_0^2-\frac{K}{2}i\omega_0\right)e^{-i\omega_0 t} - \left(-i\omega_0+\gamma-\frac{K}{2}\right)\frac{K}{2}e^{-(\gamma-\frac{K}{2})t}\right]$$

Stabilization by finite size fluctuations

Calculate stability of incoherent state to order $1/N$

Spectrum of linearized Klimontovich operator

$$\Gamma(s) = s - L$$

Obtain by solving perturbatively

$$\Gamma(s) \cdot P = \frac{1}{N}\delta(x - x')\delta(t - t')$$

Calculate $\Gamma(s)\cdot P$ using loop expansion

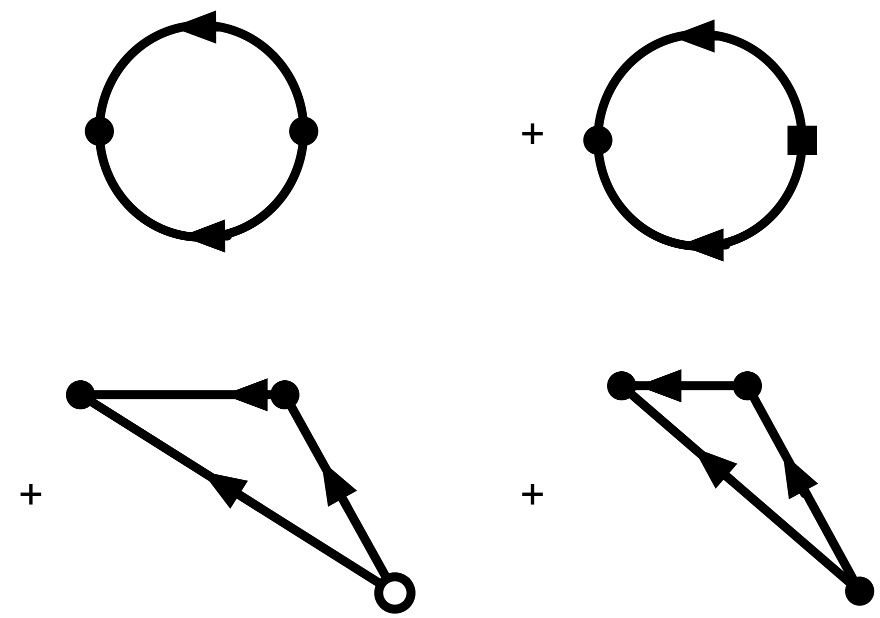
$$(\Gamma_0 + \Gamma_1) \cdot P_1 = \frac{1}{N}\delta(\theta - \theta')\delta(\omega - \omega')\delta(t - t')$$

where $\Gamma_0 \cdot P_1$ is the linearized Vlasov equation

$$\Gamma_0 \cdot P_1 = \left[\frac{\partial}{\partial t} + \omega\frac{\partial}{\partial\theta} + K\frac{\partial}{\partial\theta}\int_{-\infty}^{\infty}\int_0^{2\pi} f(\theta_1 - \theta)\rho(x_1, t)\, d\theta_1 d\omega_1\right] P_1(x, x', t - t')$$

$$+K\frac{\partial}{\partial\theta}\int_{-\infty}^{\infty}\int_0^{2\pi} f(\theta_1 - \theta)\rho(x, t) P_1(x_1, x', t - t')\, d\theta_1 d\omega_1$$

and $\Gamma_1 \cdot P_1$ is given by diagrams:



a)

c)

d)

$$\Gamma_1 \cdot P_1 = \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} d\eta \int_{t'}^{t} dt'' \Gamma_{1a}(\theta, \omega; \phi, \eta; t - t'') P_1(\phi, \eta, t''; \theta', \omega'; t')$$

$$+ \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} d\eta \int_{t'}^{t} dt' \Gamma_{1c}(\theta, \omega; \phi, \eta; t - t'') P_1(\phi, \eta, t''; \theta', \omega'; t')$$

$$+ \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} d\eta \int_{t'}^{t} dt' \Gamma_{1d}(\theta, \omega; \phi, \eta; t - t'') P_1(\phi, \eta, t''; \theta', \omega'; t')$$

where

$$\int_0^{2\pi} d\phi \int_{-\infty}^{\infty} d\nu \int_{t'}^{t} dt' \Gamma_{1a}(\theta, \omega; \phi, \nu; t - t'') P(\phi, \nu, t''; \theta', \omega'; t')$$

$$= -\frac{K^2}{N} \int d\theta_1 d\omega_1 d\theta_1' d\omega_1' d\theta_2' d\omega_2' dt_1$$

$$\frac{\partial}{\partial\theta} [f(\theta_2' - \theta) \{P_0(\theta_2', \omega_2', t; \theta_1', \omega_1', t_1) P_0(\theta, \omega, t; \theta_1, \omega_1, t_1) + P_0(\theta_2', \omega_2', t; \theta_1, \omega_1, t_1) P_0(\theta, \omega, t; \theta_1', \omega_1', t_1)\}]$$

$$\times \frac{\partial}{\partial\theta_1} [f(\theta_1' - \theta_1) \{\rho(\theta_1', \omega_1', t_1) P(\theta_1, \omega_1, t_1; \theta', \omega', t') + \rho(\theta_1, \omega_1, t_1) P(\theta_1', \omega_1', t_1; \theta', \omega', t')\}]$$

similarly for Γ_{1c} and Γ_{1d}

Results

Continuous spectrum is moved into left plane and frequency distribution is narrowed

For Cauchy distributed frequencies

$$s + in(\omega + \delta\omega) + n^2 D = 0$$

$$\delta\omega = -\frac{K^2}{2N}\frac{\omega}{\left(\gamma - \frac{K}{2}\right)^2 + \omega^2}\left[\frac{4\gamma - K}{2\gamma - K}\right] \qquad D = \frac{K^2}{2N}\frac{\gamma}{\left(\gamma - \frac{K}{2}\right)^2 + \omega^2}$$

Eigenvalue is shifted

$$s_n = -\left(\gamma - \frac{K}{2}\right) + \frac{1}{N}\frac{K}{2}\left[\left(\frac{K}{2\gamma - K}\right)\frac{K\gamma}{\left(\gamma - \frac{K}{2}\right)^2 - \gamma^2} + \frac{6\gamma - K}{2\gamma - K}\right]$$

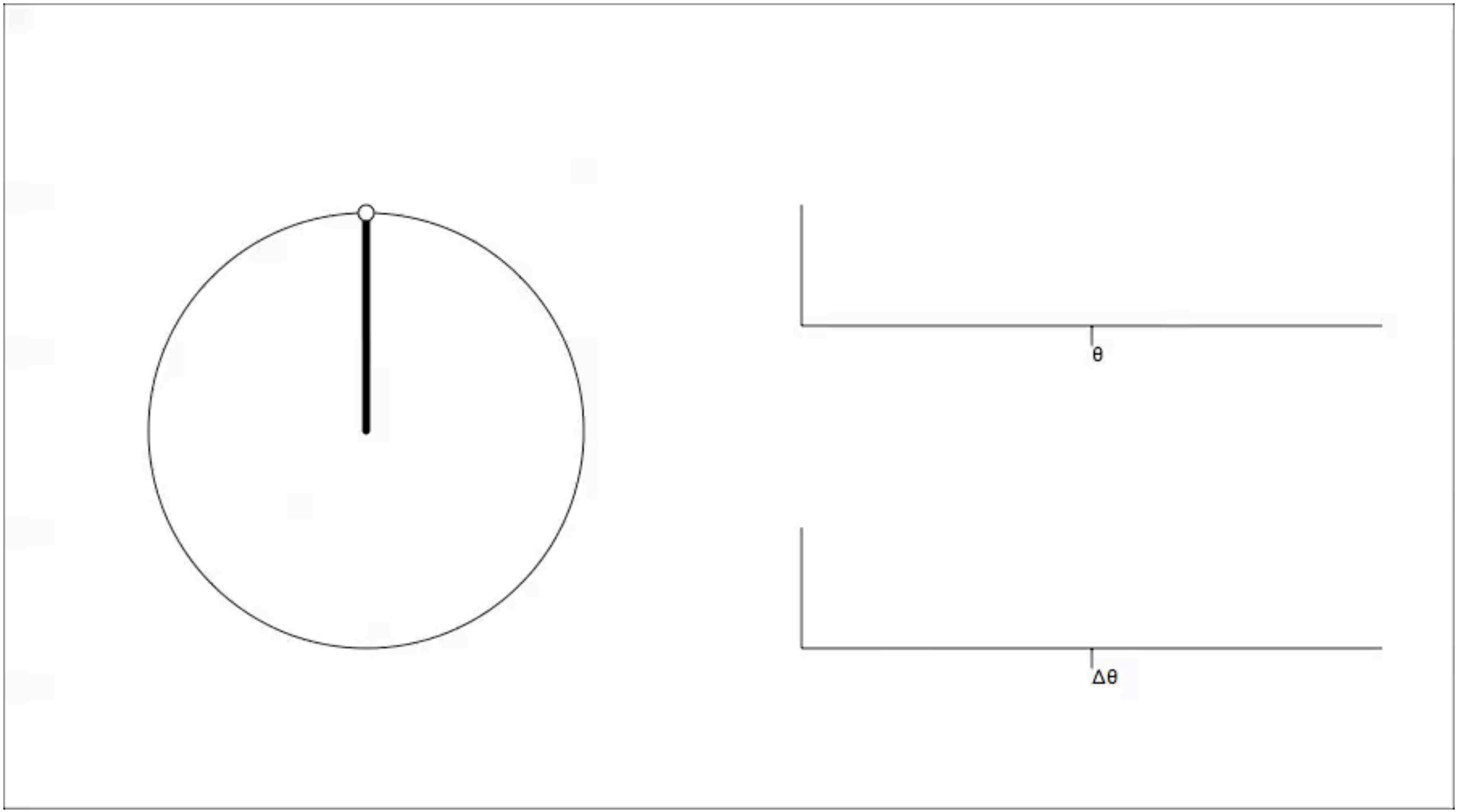
Order parameter perturbation

$$Z(t) = \frac{e^{i\theta_0}}{N}\frac{1}{\omega_0^2+\left(\gamma-\frac{K}{2}\right)^2}\left[(\gamma(\gamma-\frac{K}{2})+\omega_0^2-\frac{K}{2}i\omega_0)e^{i(\omega+\delta\omega)t-Dt}-(-i\omega_0+\gamma-\frac{K}{2})\frac{K}{2}e^{s_1 t}\right]$$

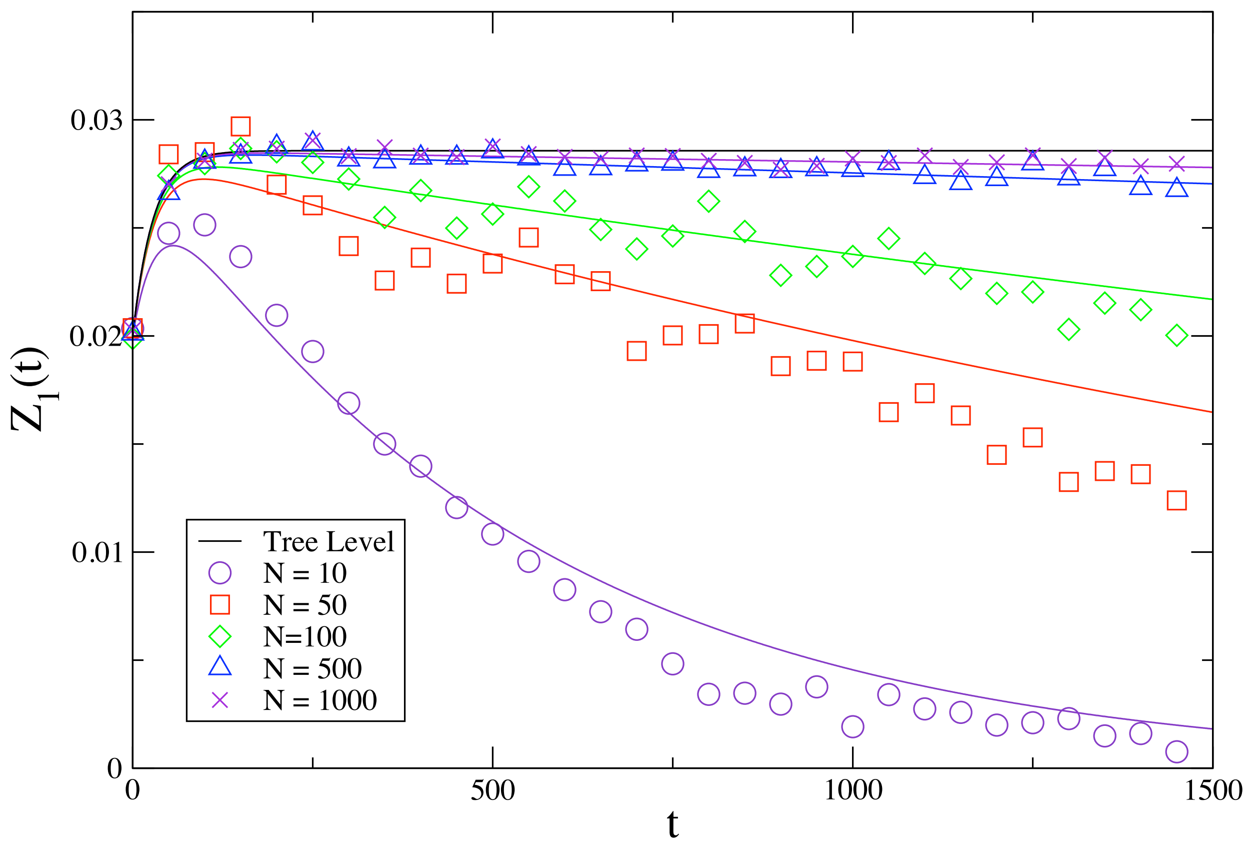
Z decays (single oscillator diffuses)

$$Z(t) = \frac{1}{N}\frac{1}{\left(\gamma-\frac{K}{2}\right)}\left[\gamma e^{-Dt}-\frac{K}{2}e^{s_1 t}\right]$$

Stability due to oscillator diffusion



Comparison to simulations

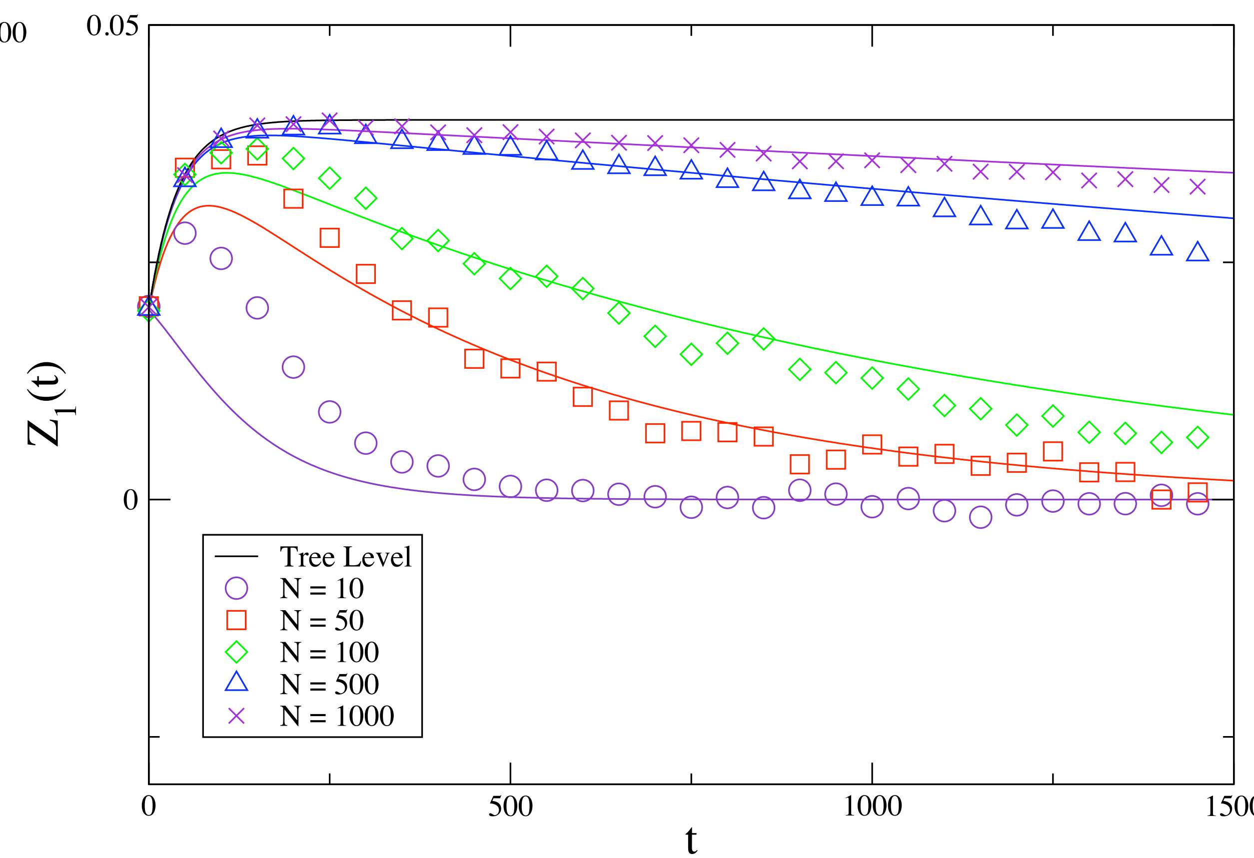


$\omega_0 = 0$

$K = 0.3K_c$

$\omega_0 = 0$

$K = 0.5K_c$



Buice and Chow, PRE (2007)

Summary

- Large but not infinite network of coupled oscillators contain correlations and fluctuations not present in mean field limit
- Kinetic theory approach (Klimontovich equation) captures these effects
- Field theory, which is equivalent to BBGKY moment hierarchy, can be used for perturbative calculations